Assignment 2 Solutions:

Note these solutions are not complete.

1. Subst $\sum_0^{\infty} a_n x^n$ into the ODE to give:

$$ (a_0 + a_1 x + a_2 x^2 + \cdots)^n - k^2 (a_0 + a_1 x + \cdots) = 0. $$

Equating coeffs of $x^n$ on both sides gives

$$ (2a_2 - k^2 a_0) = 0, \Rightarrow a_2 = k^2 a_0/2 $$

$$ (3.2a_3 - k^2 a_1) = 0, \Rightarrow a_3 = k^2 a_1/3.2 $$

$$ \cdots $$

$$(n+2)(n+1)a_{n+2} - k^2 a_n = 0, \Rightarrow a_{n+2} = k^2 a_n/((n+2)(n+1)). \quad (1) \text{ eq:1}$$

Thus $a_0, a_1$ are arbitrary and (with some effort) we get:

$$ y(x) = a_0 y_0(x) + a_1 y_1(x), \text{ where } $$

$$ y_0(x) = \sum_0^{\infty} \frac{(kx)^{2m}}{(2m)!} \equiv \cosh(kx) \text{ and } y_1(x) = \frac{1}{k} \sum_0^{\infty} \frac{(kx)^{2m+1}}{(2m+1)!} \equiv \sinh(kx)/k. $$

The expected exponentials $(\exp-(kx), \exp+(kx))$ thus arise.

(b) Note that, (see eq:1)

$$ |u_{n+2}/u_n| = |a_{n+2}/a_n| x^2 = \left| \frac{k^2 x^2}{(n+2)(n+1)} \right| \to 0 \text{ as } n \to \infty $$

for all $x$, so the series converges for all $x$ (Ratio Test).

(c) The conditions give $a_0 = 1, a_1 = -1$, so we recover the series

$$ \sum_0^{\infty} (-x)^n/n! = e^{-x}. $$

After $n$ terms the error modulus is of order

$$ E_n = \frac{|x|^{n+1}}{(n+1)!} $$

so the relative error is of order $E_n/e^{-x}$, so when $x_0 = .1$ the relative error is of order $(.1)^{n+1}/(n+1)!$: just 1 term suffices to give the required accuracy ($e^{-0.1} \approx 1$).

When $x_0 = 10$ we require

$$ \frac{10^{n+1}}{(n+1)! \times 4.5 \times 10^{-5}} < 10^{-2}; n=37 \text{ suffices; too many to be normally useful!} $$

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2. Note $x = 0$ is an ordinary point of the ODE; Taylor solutions exist for both solutions. Put

$$y = a_0 + a_1 x \cdots$$

and equate coeffs to give

$$2a_2 + a_0 = 0$$
$$3.2a_3 - 2a_2 + a_1 = 0$$

$$(n + 2)(n + 1)a_{n+2} - (n + 1)(n)a_{n+1} + a_n = 0,$$
so that $a_0, a_1$ are arbitrary and the other coefficients follow. However, it’s not easy to determine the general term of the expansion. Obtaining 3 or 4 terms is trivial by hand, and a larger number of terms is easy using M’tica. We get

$$y = a_0(1 - x^2/2 - x^3/(3.2) + \cdots) + a_1(x - 1/(3.2)x^3 + \cdots);$$

evidently the solution and its slope can be specified at $x = 0$ but the powers of $x$ don’t ‘separate out’ as in the earlier examples; this is the normal situation, not the exception. The series will converge up to the first singularity in the complex plane, and so will converge in $-1 < x < 1$. The series may converge at $x = -1$.

(b) $x = 1$ is a regular singular point of the ODE, so both solutions are of the form $(x - 1)\nu \sum_0^\infty a_n(x - 1)^n$ or some (minor) variant. One of the solutions may be Taylor in form, but certainly both won’t be. Substituting

$$y = \sum_0^\infty b_n(x - 1)^n \equiv b_0 + b_1(x - 1) + b_2(x - 1)^2 + \cdots$$

into the ODE will recover this Taylor solution, if it exists. This yields

$$b_0 = 0, \ b_2 = b_1/2, \ b_3 = b_2/3.2, \cdots, \text{ so}$$

$$y_1(x) = b_1((x - 1) + (x - 1)^2/2 + (x - 1)^3/4.3 + \cdots).$$

Note that there is just one (independent) solution of the Taylor form.
Substituting the form \(((x - 1)^\nu \sum_0^\infty a_n(x - 1)^n\) into the ODE gives

\[-a_0(\nu(\nu - 1))(x - 1)^{\nu - 1} + \text{smaller} = 0,\]

so that the indicial equation has solutions \(\nu = 0, 1\). Since the indices differ by an integer a modified form is required; the solutions are of the form (see Boyce and diPrima (p259 or p188) with \(r_1 = 1, r_2 = 0\),

\[y_1(x) = |(x - 1)|(1 + \cdots)\]

\[y_2(x) = ay_1(x)\ln |x - 1| + (1 + \cdots)\] i.e. \(y_2(x) = a|x - 1|\ln |x - 1| + 1 + \cdots\),

where \(a\) may be zero (to see one needs to carry out detailed calculations using the above form). Note that the general solution is finite as \(x \to 1\) but the derivative there is not finite (providing \(a \neq 0\)). \(x = 1\); an important result! The expansions will converge up to the next singularity in the complex plane i.e. for all \(x\).

3. (a) \(x = -3\) is a regular singular point and all other points are ordinary points of the ODE. Substituting

\[y = a_0(x + (-3))^\nu + \text{smaller}\]

gives

\[a_0(\nu(\nu - 1)(x + 3)^{\nu - 1} - 2.3\nu(x + 3)^{\nu - 1} + \text{smaller} = 0;\]

The indicial equation is

\[\nu(\nu - 1) - 6\nu = 0, \text{ i.e. } \nu(\nu - 7) = 0.\]

The indicies differ by an integer so again a modified form is necessay, see B and diP.

(b) \(x = 0, 3\) are regular singular points. Near \(x = 3\) the equation looks like

\[3(x - 3)y'' + (1 + 9)^2y' + 2.4y = 0.\]

The indicial equation is \(3\nu(\nu - 1) + 100\nu = 0\) etc.

(c) Note \(x = 0\) is an irregular singular point, \(n\pi, \ n \neq 0\ an\ integer\ are\ regular\ singularities.\ The\ behaviour\ close\ to\ \(x = 0\)\ is\ not\ prescribed\ by\ the\ regular\ singularity\ theory.\ You\ might\ try\ to\ determine\ the\ local\ behaviour.
4. The indicial equation gives $\nu = 1, 1/2$, so the solution is of the form

$$y = a_1(x + \cdots) + b_1x^{1/2}(1 + \cdots).$$

ii Evidently $y(0) = 0$ for all $a_1, b_1$ so $y(0) = 1$ can’t be satisfied. No solution.

iii $y(0) = 0$ is satisfied for all values of $a_1, b_1$. $y'(0) = 2$ can only be satisfied if $b_1 = 0$ so the only possible solution must be of the form $y = a_1(x + \cdots)$ and $a_1 = 2$ works; the solution is unique.