1 3M1 Solution: Assignment 3

1. All points except \( x = 0 \) are ordinary points so that the solution and slope can be specified at all points except \( x = 0 \) and the resulting solution is unique. Thus (a) has a unique solution, but the other problems are in doubt. One needs to determine the solution behaviour near \( x = 0 \). Note that \( x = 0 \) is a regular singular point of the ODE so solutions are of the form \( x^\nu \) near \( x = 0 \), or some ‘minor’ modification. Substituting this form into the ODE gives

\[
(\nu(\nu - 1) + \nu)x^{\nu - 1} = \text{order}(x^\nu).
\]

Thus \( \nu^2 = 0 \); equal roots so that a modified form is required, see Boyce and DiP;

\[
u_1(x) = 1 + a_1x + \cdots, \text{ where } a_1 \text{ can be determined uniquely}
\]

\[
u_2(x) = \ln|x|(1 + a_1x + \cdots) + b_1x + \cdots,
\]

with \( u \) a linear combination of these. Note that \( \nu_1(0) = 1 \) is finite, \( \nu_2(0) \) is infinite.

(b)(c) The only solutions that are finite at \( x = 0 \) are \( cu_1(x) \). If \( u(0) = 1 \) then \( c = 1 \), so that the only possible solution is \( 1 + a_1x + \cdots \). Now direct substitution of this form into the ODE and equating of coefficients of \( x^n \) gives \( a_1 = -1 \). Thus (b) has a unique solution, but there is no solution satisfying \( u'(0) = 1 \) i.e. no solution to (c).

- Note that at a singular point arbitrary (‘sensible’) initial conditions can’t normally be satisfied because at least one solution behaves badly. The flexibility normally present (two independent solutions) is thus lost and so it is unlikely there will be a unique solution. (there may be no solution or many solutions). Such issues are important (and make sense) in the physical context.

- In the above I did not make use of the knowledge that one solution of the ODE is \( e^{-x} \). Given this information I could have calculated the second solution explicitly using variation of parameters and then solved the problem exactly (rather tedious). Note, however, that knowing that one solution is \( e^{-x} \) and that the general solution is of the form \( ce^{-x} + b \ln|x| \) near \( x = 0 \) means that \( u_1 \equiv ce^{-x} \); since all other solutions are infinite at the origin. We could use this result to check out (b),(c).
• Note that in general we are not be able to ‘identify’ the finite solution (except as an infinite series).

2. (a)
   
   (i) \(u(0) = 0\) implies \(u = A \sin \lambda x\) and the second condition requires \(\lambda = n\pi\).
   
   (ii) The only solution satisfying is \(u(0) = 0\) is \(A \sinh(\lambda x)\). The remaining condition requires \(A=0\); no non-trivial solutions.
   
   (b) \(u = (1 - \cos \lambda x)/\lambda^2 + A \sin \lambda x\) satisfies the ODE with \(u(0) = 0\). Now the second bc requires
   \[
   A = \frac{1}{\sin \lambda} \left( \frac{\cos \lambda - 1}{\lambda^2} \right);
   \]
   we have a unique solution unless \(\sin \lambda = 0\), \(\lambda = n\pi\), the eigenvalues above. Now if \(\lambda = n\pi\) then there can only be a solution if the top line above vanishes, i.e. \(\cos \lambda = 1\); \(n = 0, 2, 4 \cdots\). Thus there is no solution if \(n = 1, 3, 5 \cdots\). For the other even (and 0) values of \(n\) the condition \(u(1) = 0\) is satisfied for arbitrary \(A\); the solution is non-unique.
   
   (c) In general we expect unique solutions for the non-homogeneous problem unless \(\lambda\) is an eigenvalue. If \(\lambda\) is an eigenvalue we expect no solution (usual) or many solutions.

3. The Sturm Liouville theory we developed avoided singularities because ‘anything’ can happen at a singularity; the solution could go to infinity for example, so a general theory is unavailable. However, providing the solution is reasonably behaved (so the integrals exist say), one might expect the Sturm Liouville results to go through (orthogonality etc.) because the ‘algebra’ should be almost the same. For regular singularities a theory has been developed but we did not develop this theory. (One simply looks up the appropriate text).
   
   (a) Note that \(x = 0\) is a regular singular point of the ODE so one might expect solutions to behave like \(x^\nu\) for some \(\nu\). Such solutions will be integrable if \(\nu > -1\). Now the prescribed substn gives
   \[
   \Phi'' + \lambda^2 \Phi = 0 \text{ so } \phi = \frac{\sin \lambda x}{x}, \frac{\cos \lambda x}{x}.
   \]

   One solution is thus singular at \(x = 0\), the other
   \[
   \phi_2(x) = \frac{\sin \lambda x}{x} \approx \lambda - (\lambda^3 x^2)/6 + \cdots, \text{ as } x \to 0,
   \]
so that $\phi'_2(0) = 0$. Thus $\phi_2(x)$ is both finite and has zero slope at $x = 0$ as reqd by the bc.

$\phi_2(x) = 0$ requires $\lambda = n\pi$, $n = 1, 2, \cdots$, the eigenvalues.

(b) (ii) $\phi = \frac{\sin \lambda x}{x}$.

$\phi'(1) + \phi(1) = 0$ requires

$$\sin(\lambda) + \lambda \cos(\lambda) = 0, \text{ or } \tan \lambda = -\lambda$$

Evidently there is one eigenvalue just greater than $\pi/2$ (solve numerically use Maple or M’tica etc) and then a denumerable set of eigenvalues close to $\lambda_n^0 = \frac{(2n+1)\pi}{2}$. The first few eigenvalues are best obtained using M’tica. Higher order eigenvalues need to be obtained using analysis ‘asymptotics’. Write $\lambda_n = \frac{(2n+1)\pi}{2} + \delta$ where $\delta$ is small and substitute into the defining eigenvalue equation and determine $\delta$ approximately. Thus expanding

$$\sin(\lambda_n^0 + \delta) + (\lambda_n^0 + \delta)(\cos(\lambda_n^0 + \delta)) = 0$$

for small $\delta$ gives (Taylor expansions)

$$\sin(\lambda_n^0) + \delta \cos(\lambda_n^0) + \cdots + (\lambda_n^0 + \delta)(\cos(\lambda_n^0) + \delta(-\sin(\lambda_n^0) + \cdots) = 0.$$

Noting that $\cos(\lambda_n^0) = 0$, $\sin(\lambda_n^0) = (-1)^n$ one can obtain a first estimate for $\delta$. Better estimates can be obtained by iteration.

4. Note that $x^3$ is antisymmetric, so only sine terms will appear in the Fourier expansion. Note that the periodic extension is discontinuous across $x = \pi$ so the Fourier series will converge very slowly, like $1/n$.

If we write

$$x^3 = (x^3 - \pi^2 x) + \pi^2 x \equiv h(x) + \pi^2 x,$$

and then expand $h(x)$ as a Fourier series (it’s coefficients will converge like $1/n^2$ because its PE is cts) we end up with a very accurate representation of $x^3$. The total expression can be differentiated to give $(x^3)'$ etc. Of course the best representation for $x^3$ on $(0, \pi)$ is $x^3$, but often we’re forced to use a Fourier series or minor variant.