The University of Western Australia

SECOND SEMESTER EXAMINATIONS 2004

MATHEMATICS 3A5 (539.325)
CONTINUUM MECHANICS AND INDUSTRIAL MODELLING

This paper contains: 6 pages.
5 questions.
Time allowed: Three hours
Reading time: 10 minutes

The examination consists of 5 questions. Answer all questions. Marks allocated for each question is given in brackets. The total for this examination is 100 marks.

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PLEASE NOTE

Examination candidates may only bring authorised materials into the examination room. If a supervisor finds, during the examination, that you have unauthorised material, in whatever form, in the vicinity of your desk or on your person, whether in the examination room or the toilets or en route to/from the toilets, the matter will be reported to the head of school and disciplinary action will normally be taken against you. This action may result in your being deprived of any credit for this examination or even, in some cases, for the whole unit. This will apply regardless of whether the material has been used at the time it is found.

Therefore, any candidate who has brought any unauthorised material whatsoever into the examination room should declare it to the supervisor immediately. Candidates who are uncertain whether any material is authorised should ask the supervisor for clarification.
1. Hot water flowing through a pipe of radius $a$ and length $L$ loses heat to the environment at temperature $T_0$. The water enters the pipe at a temperature $T_1$ at $x = 0$. Steady conditions are assumed. The temperature distribution in the water is uniform across a section and the temperature drop across the walls of the pipe is negligible; so $T = T(x)$ specifies temperature variations along the pipe. The heat loss rate per unit area from the pipe is assumed to be proportional to the temperature difference $T(x) - T_0$.

(a) (i) Working from first principles show that if $Q$ is the volume flux of water through the pipe $m^3/\text{sec}$, and the entry temperature of the water is $T_1$ then

$$\rho c Q \frac{\partial T}{\partial x} = -2\pi a \alpha (T(x) - T_0), \text{ with } T(0) = T_1,$$

determines temperature changes along the pipe, where $\rho, c$ are the density and specific heat of water, and $\alpha$ the heat exchange coefficient per unit surface area of pipe.

(ii) Scale the equations and thus reduce the problem to the dimensionless form

$$\frac{\partial T'}{\partial x'} = -\gamma T'(x') \text{ on } 0 < x' < 1 \text{ with } T'(0) = 1,$$

and identify $\gamma$.

(iii) Determine $T'(x')$, and comment on the results.

(10 marks)

(b) To reduce the heat loss from the water the pipe is to be insulated using a material of thickness $\delta \ll a$ and (low) conductivity $k_i$. Model the new situation and thus determine the dimensionless group that determines the effectiveness of the insulation. Comment.

*Hint* It is algebraically useful to use $T_0$ as the temperature datum.

(10 marks)

[20 marks]
2. (a) Show that the temperature distribution in semi-infinite region \( x > 0 \) due to periodic heating (so that the heat input per unit area per unit time is \( q = q_0 e^{i\omega t} \)) is given by

\[
T(x, t) = \frac{q}{\sqrt{\omega \rho c k}} e^{-(1+i)\sqrt{\omega/(2\kappa)}x} e^{i(\omega t - \pi/4)}.
\]

where \( k \) is the conductivity of the region. The heat equation is

\[
\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}.
\]

What is the length scale of penetration? (12 marks)

(b) The surface of an ‘egg shaped’ object with typical length scale \( L \) (mass \( M \), surface area \( S \), density \( \rho \), specific heat \( c \), diffusivity \( \kappa \)) is uniformly heated with a periodic input \( q_0 e^{i\omega t} \) per unit surface area per unit time. The heat loss from the surface is assumed to be negligible. The initial temperature is zero. Using the result in (a) or otherwise, determine the approximate temperature distribution for large time if

(i) \( L \gg \sqrt{\kappa/\omega} \).

(ii) \( L \ll \sqrt{\kappa/\omega} \)

Carefully explain how you arrived at your result, and indicate what ‘large time’ means in this context in both cases. (6 marks)

(iii) If the heat input rate is given by

\[ q_1 e^{i\omega_1 t} + q_2 e^{i\omega_2 t}, \text{ with } \sqrt{\kappa/\omega_1} \ll L \ll \sqrt{\kappa/\omega_2} \]

determine an approximate temperature distribution. (2 marks)

[20 marks]
3. The fundamental 3D point source solution is given by

\[ T(r, t) = H e^{-r^2/(4\kappa t)}/[8\rho c(\pi\kappa t)^{3/2}], \quad r = \sqrt{x^2 + y^2 + z^2}, \]

where \( H \) is the quantity of heat released at \( r = 0, \ t = 0 \).

(a) Write down the partial differential equation and the boundary and initial conditions satisfied by this solution and clearly indicate the dimensionality arguments that lead to this similarity solution. (Do not determine the similarity equation or its solution).

*Note:* \( \nabla^2 T(r) = T_{rr} + \frac{2}{r} T_r \) for spherical geometry.

(10 marks)

(b) Using the above similarity solution determine the temperature distribution due to an instantaneous line source of strength \( Q \) parallel to the \( z \)-axis and passing through the point \( (x', y') \) (so that the heat released per unit source length at \( t = 0 \) is \( Q \)).

*Note:* \( \int_{0}^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2} \).

(10 marks)

[20 marks]

4. The approximate time required for long iron bars (with different sectional areas) to cool from an initial temperature \( T_i \) down to the environmental temperature \( T_0 \) is required. All the bars have right-angled isosceles-triangle sections (with equal sides \( a \)).

(a) Assuming the faces of the bars are maintained at \( T_0 \), write down the equations that need to be solved to determine the cooling time. Scale the equations appropriately, and thus write down an expression that displays the cooling time dependence on \( a \) and the thermal parameters.

(8 marks)

(b) By referring to the following two problems describe (only) how a variational approach may be used to obtain an accurate estimate for the (scaled) cooling time and also clearly indicate (only) the nature of the connection between these problems.

**QUESTION 4(b) CONTINUES OVER THE PAGE**
5.

4(b) (Continued)

(i) Determine $\lambda, \phi$ such that

$$\nabla^2 \phi + \lambda^2 \phi(x,y) = 0, \text{ on } S,$$

with $\phi = 0$ around $S$.

where $S$ is defined by $x \geq 0, y \geq 0, x + y \leq 1$.

(ii) Find the $\psi(x,y)$ with

$$\psi = 0 \text{ on } \partial S, \text{ and } \int_S \psi^2 dA = 1,$$

which minimises the integral

$$\int_S (\nabla \psi)^2 dA.$$ 

(6 marks)

(c) Indicate an appropriate class of test functions to employ to obtain an approximate solution to the variational problem, and briefly indicate how an estimate for the cooling time can be obtained using such test functions.

(6 marks)

[20 marks]

SEE OVER
5. (a) The traffic equation is given by

\[ N_t + F'(N)N_x = 0. \]

Very briefly describe the basis and derivation of this equation and the procedure used to generate solutions of this equation. Also describe the traffic flow implications.

Key words: flow models, characteristics, signal speed, traffic jam.

(10 marks)

(b) A pollutant is discharged into a stream ‘at’ \( x = 0 \), and is carried downstream. The stream is moving with velocity \( u > 0 \). The pollutant is assumed to be uniformly distributed with depth and across the stream.

(i) Show that pollutant conservation requires

\[ c_t(x,t) + uc_x(x,t) = 0, \text{ in } x > 0, \]

where \( c(x,t) \) is the pollutant concentration.

(ii) Assuming the stream is initially pollutant free and a prescribed volume of pollutant is then instantaneously released at \( t = 0 \) over the interval \(-L < x < L\), so that

\[ c(x,0) = f(x) = \begin{cases} c_0 & \text{for } -L < x < L \\ 0 & \text{for } -L < x \text{ and } x > L. \end{cases} \]

Determine the solution for \( t > 0 \), using characteristics.

(10 marks)

[20 marks]