The University of Western Australia

SECOND SEMESTER EXAMINATIONS 2003

MATHEMATICS 3A5 (530.325)
CONTINUUM MECHANICS AND INDUSTRIAL MODELLING

This paper contains: 4 pages.
5 questions.
Time allowed: Three hours
Reading time: 10 minutes

The examination consists of 5 questions. Marks allocated for each question is given in brackets. The total for this examination is 105 marks.

PLEASE NOTE

Examination candidates may only bring authorised materials into the examination room. If a supervisor finds, during the examination, that you have unauthorised material, in whatever form, in the vicinity of your desk or on your person, whether in the examination room or the toilets or en route to/from the toilets, the matter will be reported to the head of school and disciplinary action will normally be taken against you. This action may result in your being deprived of any credit for this examination or even, in some cases, for the whole unit. This will apply regardless of whether the material has been used at the time it is found.

Therefore, any candidate who has brought any unauthorised material whatsoever into the examination room should declare it to the supervisor immediately. Candidates who are uncertain whether any material is authorised should ask the supervisor for clarification.
1. An infinitely long sheet of metal of negligible thickness and width $w$ intercepts radiation from the sun, receiving an input of $q_0$ per unit area of its surface per unit time. The sheet loses heat to its environment at a rate proportional to the temperature difference between its surface and its immediate environment, assumed to be at fixed temperature $T_0$. The edges of the sheet are maintained at $T_0$ and the initially plate temperature is $T_0$.

* One might expect the main body of the metal sheet to ‘quickly’ reach a temperature $T_{\text{eq}}$ determined by the local balance between radiative input and environmental loss. Close to edges one might expect a thermal adjustment to the edge temperature $T_0$. Check out these propositions using the following as a guide.

The relevant equations are

$$T_t - \kappa T_{xx} = (q_0 - \alpha (T - T_0))/(mc),$$

with $T(0, t) = T(w, t) = T_0$, and $T(x, 0) = T_0$,

where $m$ is the mass per unit area of the sheet.

(a) Determine $T_{\text{eq}}$. In terms of relevant parameters what is the time scale $t_{\text{eq}}$ required for this equilibrium to be reached? [5 marks]

(b) Scale the equations appropriately and identify the relevant dimensionless groups. It is suggested that you use the time scale $t_{\text{eq}}$, and use $T_{\text{eq}}$ as a temperature datum. [5 marks]

(c) Determine the steady state solution and sketch its profile. Under what conditions on your dimensionless parameters do the above propositions ($\star$) make sense? [10 marks]

2. The fundamental one dimensional source solution of the heat equation is given by

$$T(x, t) = \frac{Q}{2\rho c \sqrt{\pi \kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right),$$

where $Q$ is the quantity of heat released at $x = 0$, $t = 0$.

(a) Do not determine this solution, however clearly indicate the equations satisfied by this solution and the dimensionality arguments that lead to this similarity solution. [5 marks]

QUESTION 2 CONTINUES OVER THE PAGE
2 (Continued)

(b) Heat is supplied to the surface of a semi-infinite body at a rate given by \( q(t) \)
per unit surface area. Initially the body is at the temperature \( T_0 \) of its environ-
ment. Using the fundamental source solution write down expressions for
the temperature distribution within the body, and the surface temperature.
Sketch \( T(x_0, t) \) for fixed \( x_0 \) and \( t \geq 0 \)  

(c) A given amount of heat per unit area \( H \) is applied to the surface of a body
over a time interval \( \Delta \) at a constant rate.

(i) Determine \( T(x, t) \). Interpret your result.

(ii) What is the maximum temperature reached as a function of \( \Delta, H \), and
the material properties? What does this result tell us about the best
strategy to start a fire in a combustible material?  

3. (a) Show that for functions \( \phi(x) \) with \( \phi'(0) = 0 \), and \( \phi(1) = 0 \) that the functional

\[
I(\phi(x)) = \int_0^1 \left[ \phi'(x)^2 - r(x)\phi^2(x) + 2\phi(x) \right] dx
\]

is rendered stationary by the solutions \( y(x) \) of

\[
y''(x) + r(x)y(x) = 1 \quad \text{in} \quad 0 < x < 1,
\]

with \( y'(0) = 0 \) and \( y(1) = 0 \).

(b) Using this result obtain an approximate solution of the above differential
equation with boundary conditions. (Choose appropriate test functions and
write down the equations determining the coefficient/s. Do not evaluate the
integrals that arise.)

4. (a) The traffic equation is given by

\[
N_t + F'(N)N_x = 0.
\]

Very briefly describe the procedure used to generate solutions of this equation,
and also describe the traffic flow implications.

Key words: characteristics, signal speed, traffic jam.
4 (Continued)

(b) A stream of cars is lined up behind a traffic light which turns green at \( t = 0 \). The road ahead is clear. Assuming a linear speed vs. density model:

(i) Determine an explicit expression for the density distribution at time \( t = t_0 > 0 \).

(ii) Describe the traffic behaviour by referring to sketches of \( T(x, t_0) \) for a range of values of \( t_0 \).

[12 marks]

5. The insulation around an electrical cable restricts the flow of heat (generated by electricity) away from the cable into the atmosphere. If the temperature rise in the insulator is sufficiently high it may burn. A cable manufacturer would thus like to determine the dependence of the maximum insulation temperature on the heating rate and the cable parameters (thickness of insulation \( \delta \), radius of conductor \( a \), electrical heating rate \( H \) per unit volume, and thermal parameters). The insulation thickness \( \delta \ll a \) for cables of interest.

Set up a simple 1D model that you think may be of value for such a determination and carry out an initial investigation.

Note the \( \nabla^2 (T(r)) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \) in cylindricals.

[25 marks]