1. A Similarity Solution.

By using linear operator ideas we were able to determine the temperature distribution in a semi-infinite body initially at a given temperature $T_0$ when its surface is raised to (and maintained at) a different temperature $0$ for $t > 0$. It’s instructive to use similarity arguments to obtain this solution.

Firstly note that the problem is set up for the similarity procedure: the domain is semi-infinite and the initial and boundary conditions introduce no scales, apart from the temperature change at $x = 0$ for $t > 0$.

(a) Write down the equations that completely specify the problem, confirming that they are in line with the uniqueness theorem.

(b) Determine scales for the relevant quantities that reduce the number of parameters to a minimum.

(c) Show that the temperature field may be described in terms of a function $f(\eta)$ which satisfies the ordinary differential equation

$$f''(\eta) = -\eta f'(\eta)/2,$$

where $\eta$ is a suitably defined similarity variable.

(d) Show that the boundary conditions and initial conditions become

$$f(0) = 0, \quad f(\infty) = 1.$$ 

(e) Hence, using Maple, find the temperature distribution in the body and plot solution curves. Determine the heat flux at $x = 0$ and comment on the result obtained.

2. Starting a Fire?

What thermal and chemical characteristics of a material determine its combustibility, and what heating pattern is most likely to cause ignition? Ignition occurs if the temperature exceeds the ignition temperature of the material, so the problem reduces to one of determining the maximum temperature realized with different heating patterns.

A given amount of heat per unit area $H$ is applied to a surface of a body over a time interval $\Delta$. 


(a) Assuming the heating rate is uniform over the interval, use (7.29) (see notes), to show that the maximum temperature is reached at the surface of the material at the end of the heating interval. Determine this maximum value as a function of $\Delta, H$, and the material properties.

(b) Using Maple determine the maximum temperature reached as a function of the index $n$ if the rate of heating is a suitable multiple of $t^n$ for $n = -1/2, 1, 2, 3$. Compare the results with those obtained if the heating rate is uniform or if all the heat is supplied instantaneously. Interpret your results.

(c) If it is known that there is a reasonable spread of the heat supply over the time interval, what temperature rise would you predict? Estimate the amount of heat necessary to start the reaction as a function of the ignition temperature $T_{ign}$, the material properties, and $\Delta$. Comment on the material properties that are desirable for combustibility and incombustibility.

(d) What do the above results tell us is the best way to start a fire in a combustible material?

3. The Fundamental Point Source Solution

For the 3D heat equation a fundamental solution can be found by considering the instantaneous release of a quantity of heat $H$ at a point in an infinite body.

(a) The solution to this problem must be of the form $T = T(r, t, H, \rho, c, k)$ where $r = \sqrt{x^2 + y^2 + z^2}$, the distance from the release point. Why?

(b) The spherically symmetric form of the Laplace operator is

$$\nabla^2 T = T_{rr} + \frac{2}{r} T_r.$$

Write down

(i) the heat equation,
(ii) the initial conditions, and
(iii) an integral condition for the total heat content.

(c) Identify the similarity solution form.

(d) Using Maple verify that the function

$$T(r, t) = He^{-r^2/(4\kappa t)}/[8(\pi \kappa t)^{3/2}], \quad (1)$$

satisfies all the required conditions. (One can also obtain this solution by direct means by first determining the similarity equation and then using Maple to identify its solution.)

4. Lake Pollution

Phosphate from agricultural land is flushed into the upper layers of a lake of depth $h$ at $t = 0$, and then diffuses downwards. No further phosphate is flushed into the lake. The mass flux per unit area transferred downwards by turbulent diffusion is described by $m = -\kappa \partial c(z, t)/\partial z$ where $c(z, t)$ is the concentration at depth $z$ at time $t$; so the diffusion equation governs the process.
(a) Explain why, in the absence of deposition on the lake’s bottom the constraint

\[ \int_0^h c(z, t)dz = M \]

is reasonable, where \( M \) is the mass of phosphate compound per unit lake surface area flushed into the lake at \( t = 0 \).

(b) For small time it is to be expected that the fundamental one dimensional source solution will accurately describe dispersal. Write down this solution and indicate the equations this solution satisfies.

(c) Over what time scale would you expect the similarity solution to accurately describe \( c(z, t) \)?

(d) By adding to the above solution a contribution due to a second source, obtain a description that’s useful over a larger time range, and estimate the time scale for which this solution is useful. Assume no deposition.

(e) ★ Make further improvements and plot successive approximations.

(f) For large time what would you expect the concentration to be, assuming no deposition?

(g) ★★ Assuming deposition to the lake’s bottom occurs at a rate that’s proportional to the concentration there, obtain an integral equation for the deposition rate, and suggest a useful first approximation for early time.

   \textit{Hint}: Introduce an additional sink of unknown strength at the lake’s bottom, and ensure there’s no additional flux introduced at \( z = 0 \).

5. Nuclear Fallout

This problem and the problem that follows trace some of the steps involved in attempting to model the fall-out from a nuclear accident. The primary objective is to determine atmospheric concentration levels and fall-out levels at various locations. Usually the amount of material released will be unknown, so it would be necessary to infer this from measured concentration levels. A crude dispersal model which ignores fall-out is examined here.

The transport of fine nuclear material expelled into the atmosphere is dominated by air movement which may be thought of as having a steady velocity component (with non-zero mean) with turbulent fluctuations (with zero mean) superimposed. Basically the mean flow carries the nuclear material with it, while the fluctuations cause mixing and thus dispersal of the material. It may be shown theoretically and displayed experimentally that the rate and direction of dispersal of a material due to turbulent fluctuations depends on the concentration gradient, so the situation is analogous to heat dispersal with the mass flux per unit area given by \( m = -\kappa \nabla c \) where \( c \) is the concentration of material and the \textit{dispersion coefficient}, \( \kappa \), will depend on the size of the turbulent fluctuations; experimental data is available.

We’ll assume the mean velocity and the dispersion coefficient remain fixed, and we’ll use a co-ordinate system moving with the mean flow. We’ll examine the situation in
which the accident releases a mass $M$ of material at $t = 0$ at $(x, y, z) = (0, 0, 0)$, with $0 < z < h$, where $h$ is the effective height of the atmosphere. After this release the leak is plugged.

In the initial stages the dispersal process will be very complicated and dependent on the details of the accident. However, after a time scale of order $h^2/\kappa$, one might expect the concentration profile in the vertical direction to settle down; for simplicity let’s assume for this preliminary model that the concentration is independent of $z$, so $c(x, y, z, t) \equiv c(x, y, t)$; and that the fall-out is negligible.

Under the prescribed circumstances $c(x, y, z) \equiv c(r, t)$ where $r = \sqrt{x^2 + y^2}$ (why?), so that it’s appropriate to use the cylindrical form of the Laplacian and the dispersal is governed by

$$\frac{\partial c}{\partial t} = \kappa \left( \frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} \right).$$

(a) Use scaling arguments to show that the similarity solution form is given by

$$c(r, t) = \frac{M}{\kappa ht} C(\xi)$$

where $\xi = r^2/\kappa t$, and determine the equation for $C$.

(b) Use Maple to obtain the similarity solution explicitly.

(c) Plot $C(\xi)$ and comment on the solution behaviour.

(d) Plot $c(r_0, t)$ for various distances $r_0$ measured from the origin in the moving frame (ie. the effective source location). Comment.

(e) Plot the maximum (scaled) concentration levels expected at various distances $r_0$ from the effective source location, and determine the expected time for concentration peaks. A typical figure for $\kappa$ is $5 \times 10^4 \text{cm}^2/\text{s}$ and for mean wind speed, 15m/sec.

(f) Although from the scientist’s point of view the above plots are of most interest, civil authorities would like to know about the changes in concentration levels at fixed locations on the earth. Plot scaled concentration levels at various scaled locations downstream from the accident as a function of scaled time.