1. Starting a Fire?

What thermal and chemical characteristics of a material determine its combustibility, and what heating pattern is most likely to cause ignition? Ignition occurs if the temperature exceeds the *ignition temperature* of the material, so the problem reduces to one of determining the maximum temperature realized with different heating patterns.

A given amount of heat per unit area $H$ is applied to a surface of a body over a time interval $\Delta$.

(a) Assuming the heating rate is uniform over the interval, use (7.29) (see notes), to show that the maximum temperature is reached at the surface of the material at the end of the heating interval. Determine this maximum value as a function of $\Delta, H$, and the material properties.

(b) Using Maple determine the maximum temperature reached as a function of the index $n$ if the rate of heating is a suitable multiple of $t^n$ for $n = -1/2, 1, 2, 3$. Compare the results with those obtained if the heating rate is uniform or if all the heat is supplied instantaneously. Interpret your results.

(c) If it is known that there is a reasonable spread of the heat supply over the time interval, what temperature rise would you predict? Estimate the amount of heat necessary to start the reaction as a function of the ignition temperature $T_{ign}$, the material properties, and $\Delta$. Comment on the material properties that are desirable for combustibility and incombustibility.

(d) What do the above results tell us is the best way to start a fire in a combustible material?

2. The Fundamental Point Source Solution

For the 3D heat equation a fundamental solution can be found by considering the instantaneous release of a quantity of heat $H$ at a point in an infinite body.

(a) The solution to this problem must be of the form $T = T(r, t, H, \rho, c, k)$ where $r = \sqrt{x^2 + y^2 + z^2}$, the distance from the release point. Why?
(b) The spherically symmetric form of the Laplace operator is

\[ \nabla^2 T = T_{rr} + \frac{2}{r} T_r. \]

Write down

(i) the heat equation,
(ii) the initial conditions, and
(iii) an integral condition for the total heat content.

(c) Identify the similarity solution form.

(d) Using Maple verify that the function

\[ T(r, t) = \frac{He^{-r^2/(4\kappa t)}}{[8(\pi \kappa t)^{3/2}]}, \]  

satisfies all the required conditions. (One can also obtain this solution by direct means by first determining the similarity equation and then using Maple to identify its solution.)

3. Lake Pollution

Phosphate from agricultural land is flushed into the upper layers of a lake of depth \( h \) at \( t = 0 \), and then diffuses downwards. No further phosphate is flushed into the lake. The mass flux per unit area transferred downwards by turbulent diffusion is described by \( m = -\kappa \partial c(z, t)/\partial z \) where \( c(z, t) \) is the concentration at depth \( z \) at time \( t \); so the diffusion equation governs the process.

(a) Explain why, in the absence of deposition on the lake’s bottom the constraint

\[ \int_0^h c(z, t)dz = M \]

is reasonable, where \( M \) is the mass of phosphate compound per unit lake surface area flushed into the lake at \( t = 0 \).

(b) For small time it is to be expected that the fundamental one dimensional source solution will accurately describe dispersal. Write down this solution and indicate the equations this solution satisfies.

(c) Over what time scale would you expect the similarity solution to accurately describe \( c(z, t) \)?

(d) By adding to the above solution a contribution due to a second source, obtain a description that’s useful over a larger time range, and estimate the time scale for which this solution is useful. Assume no deposition.

(e) ★ Make further improvements and plot successive approximations.

(f) For large time what would you expect the concentration to be, assuming no deposition?
(g) ** Assuming deposition to the lake’s bottom occurs at a rate that’s proportional to the concentration there, obtain an integral equation for the deposition rate, and suggest a useful first approximation for early time.

Hint: Introduce an additional sink of unknown strength at the lake’s bottom, and ensure there’s no additional flux introduced at $z = 0$.

4. Nuclear Fallout

This problem and the problem that follows trace some of the steps involved in attempting to model the fall-out from a nuclear accident. The primary objective is to determine atmospheric concentration levels and fall-out levels at various locations. Usually the amount of material released will be unknown, so it would be necessary to infer this from measured concentration levels. A crude dispersal model which ignores fall-out is examined here.

The transport of fine nuclear material expelled into the atmosphere is dominated by air movement which may be thought of as having a steady velocity component (with non-zero mean) with turbulent fluctuations (with zero mean) superimposed. Basically the mean flow carries the nuclear material with it, while the fluctuations cause mixing and thus dispersal of the material. It may be shown theoretically and displayed experimentally that the rate and direction of dispersal of a material due to turbulent fluctuations depends on the concentration gradient, so the situation is analogous to heat dispersal with the mass flux per unit area given by $m = -\kappa \nabla c$ where $c$ is the concentration of material and the dispersion coefficient, $\kappa$, will depend on the size of the turbulent fluctuations; experimental data is available.

We’ll assume the mean velocity and the dispersion coefficient remain fixed, and we’ll use a co-ordinate system moving with the mean flow. We’ll examine the situation in which the accident releases a mass $M$ of material at $t = 0$ at $(x, y, z) = (0, 0, 0)$, with $0 < z < h$, where $h$ is the effective height of the atmosphere. After this release the leak is plugged.

In the initial stages the dispersal process will be very complicated and dependent on the details of the accident. However, after a time scale of order $h^2/\kappa$, one might expect the concentration profile in the vertical direction to settle down; for simplicity let’s assume for this preliminary model that the concentration is independent of $z$, so $c(x, y, z, t) \equiv c(x, y, t)$; and that the fall-out is negligible.

Under the prescribed circumstances $c(x, y, z) \equiv c(r, t)$ where $r = \sqrt{x^2 + y^2}$ (why?), so that it’s appropriate to use the cylindrical form of the Laplacian and the dispersal is governed by

$$\frac{\partial c}{\partial t} = \kappa \left( \frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} \right).$$

(a) Use scaling arguments to show that the similarity solution form is given by

$$c(r, t) = \frac{M}{\kappa h t} C(\xi)$$

where $\xi = r^2/\kappa t$, and determine the equation for $C$. 

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(b) Use Maple to obtain the similarity solution explicitly.

(c) Plot $C(\xi)$ and comment on the solution behaviour.

(d) Plot $c(r_0, t)$ for various distances $r_0$ measured from the origin in the moving frame (ie. the effective source location). Comment.

(e) Plot the maximum (scaled) concentration levels expected at various distances $r_0$ from the effective source location, and determine the expected time for concentration peaks. A typical figure for $\kappa$ is $5 \times 10^4 \text{cm}^2/\text{s}$ and for mean wind speed, $15\text{m/sec}$.

(f) Although from the scientist’s point of view the above plots are of most interest, civil authorities would like to know about the changes in concentration levels at fixed locations on the earth. Plot scaled concentration levels at various scaled locations downstream from the accident as a function of scaled time.

5. Peel Inlet Pollution Problems I

Excessive algal growth in the Peel Inlet in Western Australia has caused damage to estuarine marine life, and an unpleasant stench (of rotting algae) for residents. As you can well imagine the loss to the tourist industry is also considerable. The environmental imbalance arises basically because, in an attempt to increase production levels, farmers in the area have used excessive fertilizer levels. Rain and irrigation water carry the excess superphosphate into the groundwater below, which subsequently flows into the estuary. The build up in phosphate leads to algal blooms. The problem is basically a political/social/economic one (how to persuade farmers to use less fertilizer) but the modeller can contribute by determining concentration levels, time spans, and by commenting on the effectiveness of possible remedies.

**Background:** Moist soil consists of a mixture of soil particles, air and water. Under saturated conditions water fills the voids not occupied by solid particles. In regions in which the soil overlays impermeable rock (typical of catchment areas) rainfall and irrigation water (together with soluble contaminants) percolate through the soil to the saturated groundwater below which is gravitationally driven to the ocean. The velocities of flow (defined to be the volume of water crossing unit area in unit time) are small; of the order of metres per month.

Here we model the groundwater transport aspect of the problem. We’ll assume

- the groundwater flow is one-dimensional and uniform,
- the phosphate is carried along with the groundwater,
- no phosphate is absorbed by soil particles and it moves without diffusion.

The phosphate concentration $c(x, t)$ is measured in kg/m$^3$ of groundwater. A known quantity of phosphate per unit area per unit time $Q \text{kg/m}^2/\text{s}$ is released into the groundwater at $x = 0$ for $t > 0$, see Fig. ??.

(a) Show that phosphate conservation requires

$$c_t(x, t) + uc_x(x, t) = 0, \quad x > 0,$$
where \( u \) is the groundwater flow velocity, and that the conditions envisaged above require
\[
c(0, t) = Q/u, \quad c(x, 0) = 0.
\]

(b) Find the solution using characteristics.

(c) Using the typical value \( u = 10^{-4} \text{m/s} \) (for sand) determine how long it would take for the phosphate to reach the Inlet (distance 2 km from the source) and the concentration levels that are to be expected there.

6. Peel Inlet Pollution Problems II

One might expect diffusive effects of various types to simply smooth out the step function concentration profile obtained in the previous exercise. Thus one might expect a profile of the form
\[
c(x, t) = C(\xi, t), \quad \text{where} \quad \xi = x - ut
\]
where \( C \) is “close” to the step function solution obtained above. To check this out we’ll examine the model equation
\[
c_t + uc_x = \kappa c_{xx},
\]
in which a diffusion type term has been added.

(a) Show that if \(-\kappa c_x\) is the diffusive flux of phosphate, then the above equation does in fact represent an accurate description of the phosphate conservation equation. In practice \( \kappa \) would be experimentally determined, and would be \( u \) dependent.

\textit{Hint:} Examine a volume element moving with the groundwater.

(b) Substitute the above form into the model equation and thus show that a solution of the expected form exists if \( C \) satisfies
\[
C_t = \kappa C_{\xi\xi}.
\]

(c) What initial conditions on \( C \) are appropriate? Determine \( C \).

7. Pollutant Dispersal ***

In addition to the convective and diffusive effects discussed in the last two exercises, chemical or biological decay and/or deposition usually play a role in determining the concentration levels of pollutants in the environment. Often the assumption is made that the loss rate is proportional to the concentration. We’ll examine the case of pollutant dispersal in a river flowing with velocity \( u \). The pollutant will be assumed to be uniformly distributed with depth and across the river. Diffusive effects will be ignored in this one-dimensional model.

(a) Show that pollutant conservation is assured if
\[
c_t(x, t) + uc_x(x, t) = -\alpha c(x, t),
\]
where \( \alpha c \) is the loss rate per unit volume of water if the concentration of pollutant is \( c \).
(b) We’ll assume the river is initially pollutant free and a prescribed volume of pollutant is then instantaneously released at $t = 0$ over the interval $-L < x < L$, so that

$$c(x, 0) = f(x) = \begin{cases} c_0 & -L < x < L \\ 0 & x < -L, \text{ and } x > L. \end{cases}$$

Using characteristics determine the solution in the $\alpha = 0$ case.

(c) In the $\alpha \neq 0$ case, given the equation is linear, one might expect an exponential decay in pollutant levels so it makes sense to look for a solution of the form

$$e^{-\lambda t}C(x, t)$$

By choosing $\lambda$ appropriately remove the “sink” term from the equation for $C$ and thus show that the solution for $c$ is given by

$$c(x, t) = f(x - ut)e^{-\alpha t}.$$ 

You’ll recall that this is a trick often used to simplify ordinary differential equations. Interpret this result.

(d) $*$ Suppose now that we have a situation in which the pollutant release rate at $x = 0$ is specified by

$$r(t) = \begin{cases} r_0 & 0 < t < T_0 \\ 0 & t > T_0. \end{cases}$$

Determine concentration levels downstream from the release point. How far downstream does one need to go for concentration levels to drop to 1% of those at the source?

*Hint:* Modify the trick used in (c) to obtain an appropriate solution form for this situation.