Asymptotic Solution of the Helmholtz Equation.

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1 Introduction

1.1

This work is a study of approximations to solutions to a class of problem for the Helmholtz (reduced wave) equation

\[ \nabla^2 H + \left( \frac{N(r)}{\epsilon} \right)^2 H = 0, \quad 0 < \epsilon \ll 1 \quad (1.1.1) \]

on \( \mathbb{R}^3 \). These (effectively initial value) problems have \( H \) and its normal derivative prescribed on a suitable initial surface \( S \), points of which are

\[ r = S(\psi, \theta). \quad (1.1.2) \]

Fundamental assumptions are that \( N \) is analytic, positive and bounded everywhere, and nowhere small; while the components of \( S \) are everywhere analytic. Further \( S \) is supposed to have principal curvature radii of the same sign and, in the constant-\( N \) case, not to re-intersect its normals. (When \( N \) is not constant, the assumption is that the geometric optic rays from points in \( S \) do not re-intersect.) More assumptions will be introduced as required, or may be implicit.

Approximate solutions to the above stated problem are classically constructed using the method of geometrical optics [4], with modification at its singularities. A different approach is taken below, which will confirm and extend the classical results. A foundation of this discussion in the constant-\( N \) case is the use of principal coordinates \( (\psi, \theta) \) to describe \( S \) (section 2.1), and their calculation is shown in section 2.2. When \( N(r) \) is not constant, the computational advantage of the principal coordinate choice is not so compelling, but their use simplifies the arguments.

With a principal coordinate system defined on \( S \), there is a natural local orthogonal coordinatization of the adjacent region in which a field point \( r \) is

\[ r(\psi, \theta, \sigma) = (S + \sigma n)(\psi, \theta). \quad (1.1.3) \]

Here \( n(\psi, \theta) \) is the local unit normal to \( S \), directed towards the latter’s concavity and

\[ \sigma = \pm \min(\psi, \theta)_{r \text{constant}} |r - S(\psi, \theta)| \]

\[ = \pm |r(\psi, \theta, \sigma) - S(\psi, \theta)| \quad (1.1.4) \]

is positive on the concave side.

This choice of coordinates leads to an examination of (what are in the constant-\( N \) problem) caustic surfaces (section 2.1). These are surfaces on which the coordinate system \( (1.1.3) \) becomes singular: for these coordinates \( r_1(\psi, \theta) \) on which

\[ r_\psi = 0 \quad (\sigma = R_1(\psi, \theta)) \quad (1.1.5) \]

and \( r_2(\psi, \theta) \) on which

\[ r_\theta = 0 \quad (\sigma = R_2(\psi, \theta)). \quad (1.1.6) \]
Even for simple but non-degenerate examples, caustic surface geometry can be complicated. In the one studied below (section 2.3), there are two intersecting caustic surfaces, and each surface has at least one singular line on which only one surface tangent vector exists. (These lines are the analogues of the cuspidal points on the evolute of a plane curve associated with local extreme values of the latter’s radius of curvature function.) The singular lines also have singular points at which no surface tangent vector exists.

The geometry of, for example, the caustic surface $r_1$ at a non-singular point $r_1(\psi_0, \theta_0) = p$ on it is related to the geometry of $S$ at $S(\psi_0, \theta_0)$, and by extension to that of the coordinate surfaces $\sigma = \sigma_0$ ($0 < \sigma_0 < R_1(\psi_0, \theta_0)$) at $r(\psi_0, \theta_0, \sigma_0)$. The tangent plane to the coordinate surface $\theta = \theta_0$ when $\psi = \psi_0$ intersects the caustic surface $r_1$ in a plane curve which contains the point $p$. If the radius of curvature of this plane curve is $\mu$, then $\mu$ is connected to the geometry of $S$ and it is well known (or easily shown) that

$$\mu(p) = \left( \frac{R_1}{|S_\psi|} \right) (\psi_0, \theta_0). \quad (1.1.7)$$

In section 2.2 it is shown that the principal directions on the coordinate surfaces $\sigma = \sigma_0$ at $r(\psi_0, \theta_0, \sigma_0)$ are parallel to $S_\psi(\psi_0, \theta_0)$ and $S_\theta(\psi_0, \theta_0)$. Further, if a scalar field $R_1(\psi, \theta, \sigma)$ is defined for coordinate surfaces $\sigma = \sigma_0$ as the principal curvature radius for the direction $S_\psi$, then equation (1.1.7) can be restated as

$$\mu(p) = \left( \frac{R_1}{|S_\psi|} \right) (\psi_0, \theta_0, \sigma) \cdot 0 < \sigma < R_1(\psi_0, \theta_0). \quad (1.1.8)$$

The $R_1$ field is related to the $R_1$ field through

$$R_1(\psi_0, \theta_0, \sigma_0) = \frac{R_1(\psi_0, \theta_0)}{\sigma_0} \quad (1.1.9)$$

and so

$$R_1(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) = 0. \quad (1.1.10)$$

Thus the coordinate surface $\sigma = R_1(\psi_0, \theta_0)$ is singular at the point in $r(\psi_0, \theta_0, R_1(\psi_0, \theta_0))$ because its principal curvature radius $R_1$ vanishes there, and it is this singularity which, in a qualitative sense, allows the coordinate surface to terminate at this point - as it will at all other points $r(\psi', \theta', R_1(\psi', \theta'))$ such that $R_1(\psi', \theta') = R_1(\psi', \theta')$.

It is also shown in section (2.2) that, if $\nu(\psi_0, \theta_0)$ is a unit normal to $r_1$ at $r(\psi_0, \theta_0, R_1(\psi_0, \theta_0))$, then

$$\nu(\psi_0, \theta_0) = \pm \left( \frac{S_\psi}{|S_\psi|} \right) (\psi_0, \theta_0). \quad (1.1.11)$$

It follows that the limit form of the result (1.1.8) can be expressed as

$$\lim_{\sigma \to R_1(\psi_0, \theta_0)} \left( \frac{R_1}{|\nu \cdot \nabla R_1|} \right) = \mu(p). \quad (1.1.12)$$
Natural formulae analogous to (1.1.7-12) can be stated for the caustic surface \( r_2 \), with \( \theta \)-partial derivatives replacing \( \psi \)-partial derivatives, \( R_2 \) replacing \( R_1 \), and an appropriate radius of curvature defined in place of \( \mu \).

The results which correspond to (1.1.11-12) for geometrical optics with \( N(r) \) not constant (section 4.3) are central to achieving the matching procedure described in section 4.4.

In section 3 an integral solution of equation (1.1.1) with

\[
H(r) = (2\pi \epsilon)^{-1} \int_{V} \frac{K(\psi) \exp(i|\psi - S(r)|/\epsilon)}{|\psi - S(r)|} d\psi
\]

is studied using variants of, and extensions of the method of stationary phase [6]. These amount to the proposition: if \( q_1 \) are asymptotically positive quartic polynomials, not linear but with some suitable coefficients zero, the integral,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( i(\epsilon q_1(\xi) + q_2(\eta))(1 + \delta Q(\xi, \eta)) \right) d\xi d\eta,
\]

which is assumed to exist, can be asymptotically approximated by

\[
\exp \left( i(\epsilon q_1(0) + q_2(0)) \right) \int_{-\infty}^{\infty} \exp \left( i(\epsilon q_1(\xi) - q_1(0)) \right) d\xi \int_{-\infty}^{\infty} \exp \left( i(\epsilon q_2(\eta) - q_2(0)) \right) d\eta
\]

as \( \delta \to 0 \), if \( Q(0,0)=0 \) and \( Q \) is suitably behaved (uniformly bounded, and with a uniformly bounded variation property for the real and imaginary parts of its analytic continuations in both variables) otherwise.

In section (3.1) the results of geometrical optics are reproduced for field points \( r \) not near the caustic surfaces generated by \( S \). This (extended) stationary phase calculation identifies the geometric optics rays as normals to \( S \) which pass through the field point, and gives the usual geometric optics approximation the \( k \)th ray passing through the field point

\[
L_k(r) \approx L_k \cos(\phi_k/\epsilon)
\]

for points on it that are not near points its points of tangency with caustic surfaces. The geometric optics approximation to (1.1.13) is the sum of approximations of the type (1.1.14) arising from all rays passing through the field point.

In equation (1.1.14), \( A_k \) is the 'amplitude' and \( \phi_k/\epsilon \) is the 'phase' of the right hand side term; also

\[
\phi_k(r) = |r - S(\phi_k, \theta_k)|.
\]

where \((\phi_k, \theta_k)\) identifies the particular ray.

If a given ray's point of tangency with a caustic surface is not a singular point of the latter, a variant of the method of stationary phase (section 3.2) can be used to approximately evaluate the integral (1.1.13) for field points \( r \) on the normal to the caustic surface at the ray's point of tangency, and near it. This calculation produces results which properly match those of section 3.1 as the distance from the caustic surface becomes non-small.
Finally, in section 3.3, a further variation of the stationary phase method is used to study the approximation of the integral for points on a critical ray passing through a point on a singular line of just one of the caustic surfaces.

In section 4 the more general geometric optics arising when \( N \) is not constant is discussed. The most convenient form of the basic equations is stated in section 4.1, and their integration is examined in section 4.2. Some properties of their solutions are extracted in section 4.3, most particularly those of the constant phase surfaces at their singular termination on caustic surfaces. These properties are analogues of the properties (1.1.11-12) summarised earlier. Their importance is, as before, that they identify parameters in a near caustic, local approximate solution of equation (1.1.1) (that is, the local approximate evaluation of the integral (1.1.13) obtained in section 3.2) that can be used to connect the geometric optics solutions which apply to a ray on either side of its point of tangency at a regular point of a caustic surface.

2 Geometrical Considerations

2.1

The key to this theory is an optimal choice of coordinates in which to describe approximate solutions of the Helmholtz problem (1.1.1-2) in the constant-\( N \) case. Then two of the coordinates (\( \psi \) and \( \theta \)) are principal coordinates on \( S \)

\[
\mathbf{r} = S(\psi, \theta),
\]

so that tangent vectors to coordinate lines \( \psi = \text{constant} \) and \( \theta = \text{constant} \) in that surface are in principal directions at each point on it, and thus orthogonal at each point.

Other consequences of this choice are the following. Suppose \( \mathbf{n}(\psi, \theta) \) is the local unit normal vector to \( S \), and the scaling functions

\[
\begin{align*}
h_1(\psi, \theta) &= |S_\psi|(\psi, \theta) \\
h_2(\psi, \theta) &= |S_\theta|(\psi, \theta)
\end{align*}
\]

are known, thereby completing the definition of the coordinate system. Then the normal component of the mixed partial derivative \( S_{\psi \theta} \) satisfies

\[
S_{\psi \theta} \cdot \mathbf{n} = 0
\]

everywhere, and conversely global satisfaction of this condition ensures that the coordinates are principal. Further, at any point on \( S \) the vectors \( \mathbf{n}_\psi \) and \( \mathbf{S}_\psi \) are parallel: it is usual to write

\[
\mathbf{n}_\psi = -\mathbf{S}_\psi / R_1
\]

where \( R_1 \) is one of the local principal curvature radii. It follows that

\[
R_1 = h_1^2 / (S_{\psi \psi} \cdot \mathbf{n})
\]
everywhere. Similar relations connect the other derivatives:

\[ n_\theta = -S_{\theta}/R_2 \]  \hfill (2.1.6)

where

\[ R_2 = h_2^2/(S_{\theta\theta} \bullet n) \]  \hfill (2.1.7)

is the other principal curvature radius. (Formulae such as

\[ n \bullet \frac{\partial}{\partial \psi} (S_\psi/h_1) = h_1/R_1, \]

\[ S_{\psi\psi} = (h_1/\psi) S_\psi - (h_1 h_{1\psi}/h_1^2) S_\theta + (h_1^2/R_1)n, \]

and

\[ S_{\psi\theta} = (h_2/\psi) S_\theta + (h_1 h_{1\theta}/h_1) S_\psi \]

may be useful.)

It follows that there is a natural local coordinatization of \( \mathbb{R}^3 \) with coordinate surfaces \( \psi = \text{constant} \) and \( \theta = \text{constant} \) generated by normals to \( S \), so that points \( r \) can be located as

\[ r(\psi, \theta, \sigma) = S(\psi, \theta) + \sigma n(\psi, \theta). \]  \hfill (2.1.8)

This coordinatization fails when either

\[ r_\psi = 0 \quad \text{or} \quad r_\theta = 0, \]

that is when

\[ \sigma = R_1 \quad \text{or} \quad \sigma = R_2, \]  \hfill (2.1.9)

following from the properties (2.1.4) and (2.1.6). The surfaces

\[ r_j(\psi, \theta) = S(\psi, \theta) + R_j n(\psi, \theta) \quad j = 1, 2 \]  \hfill (2.1.10)

therefore envelopes of normals to \( S \), and are called caustic surfaces. All surfaces \( \sigma = \text{constant} \) (2.1.8) will be called coordinate surfaces.

Consider any coordinate surface \( \sigma = \sigma_0 \)

\[ r(\psi, \theta, \sigma_0) = S(\psi, \theta) + \sigma_0 n(\psi, \theta). \]  \hfill (2.1.11)

It has principal directions on it parallel to \( S_\psi \) and \( S_\theta \) at \( (\psi, \theta) \), because of the everywhere vanishing scalar product

\[ n \bullet r_{\psi\theta} = \left( n \bullet S_{\psi\theta} + \sigma_0 n \bullet \left( \frac{S_\psi R_{1\theta}}{R_2^2} - \frac{S_{\theta\phi}}{R_1} \right) \right)(\psi, \theta, \sigma_0) \]

\[ = 0, \]  \hfill (2.1.12)

using the converse of the property (2.1.3). The principal curvature radii of the surface \( \sigma = \sigma_0 \) are \( (R_1(\psi, \theta) - \sigma_0) \) and \( (R_2(\psi, \theta) - \sigma_0) \) respectively at a point \( r(\psi, \theta, \sigma_0) \) in it.
Now examine a particular, singular member of the family of coordinate surfaces
\[ \mathbf{r}(\psi, \theta, R_1(\psi_0, \theta_0)) = S(\psi, \theta) + R_1(\psi_0, \theta_0)\mathbf{n}(\psi, \theta) \] (2.1.13)
in the neighbourhood of a point
\[ \mathbf{r}(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) = S(\psi_0, \theta_0) + R_1(\psi_0, \theta_0)\mathbf{n}(\psi_0, \theta_0) \] (2.1.14)
in common with the caustic surface \( r_1 \) (2.1.10). At this point the coordinate surface \( \sigma = R_1(\psi_0, \theta_0) \) has zero principal curvature radius corresponding to the \( S_\psi \) direction. (In a qualitative sense, the vanishing principal curvature radius is the singularity which allows the coordinate surface \( \sigma = R_1(\psi_0, \theta_0) \) to terminate on the caustic surface \( r_1 \).) Moreover the \( S_\sigma \) direction at the common point is also normal to the caustic surface. This follows because two tangent vectors to the caustic surface are
\[ \mathbf{r}_1^\prime(\psi_0, \theta_0) = (R_1^\prime \mathbf{n})(\psi_0, \theta_0) \] (2.1.15)
if, as is now assumed, \( R_1^\prime(\psi_0, \theta_0) \neq 0 \) and
\[ \mathbf{r}_1(\psi_0, \theta_0) = (S_\theta(1 - R_1/R_2) + R_1 \mathbf{n})(\psi_0, \theta_0), \] (2.1.16)
so a vector normal to the caustic surface \( r_1 \) at \( (\psi_0, \theta_0, R_1(\psi_0, \theta_0)) \) is
\[ (\mathbf{r}_1^\prime \times \mathbf{r}_1) (\psi_0, \theta_0) \text{ or } (\mathbf{n} \times S_\theta) \]
that is, parallel to the principal direction \( S_\psi(\psi_0, \theta_0) \), since \( S_\psi, S_\theta, \) and \( \mathbf{n} \) are mutually orthogonal at any particular point of \( S \). This result will hold for any point \( \mathbf{r}(\psi, \theta, R_1(\psi, \theta)) \) common to \( r_1(\psi, \theta) \) and the coordinate surface \( \sigma = R_1(\psi, \theta) \), providing, as is assumed, \( R_1^\prime \neq 0 \) and \( R_1 \neq R_2 \) there. (If \( R_1^\prime(\psi', \theta') = 0 \), there is no uniquely defined normal to the caustic surface - \( \mathbf{r}(\psi', \theta', R_1(\psi', \theta')) \) lies on a singular line of the caustic surface; if \( R_1(\psi'', \theta'') = R_2(\psi'', \theta'') \) then \( \mathbf{r} = S(\psi'', \theta'') \) is an umbilical point on the initial surface and the normal direction on the caustic surface at \( \mathbf{r} = r_1(\psi'', \theta'') \) is again not uniquely defined.)
Thus, save for the excepted points, at a point common to a coordinate surface and the associated caustic surface, the normal to the caustic surface is also \( (\pm) \) the principal direction corresponding to zero curvature radius on the coordinate surface.

Some geometrical parameters of the caustic and initial surfaces are connected at \( (\psi_0, \theta_0) \). Consider the tangent plane coordinate surface \( \theta = \theta_0 \) at \( \psi = \psi_0 \). It contains the line
\[ \mathbf{r}(\psi_0, \theta_0, \sigma) = S(\psi_0, \theta_0) + \sigma \mathbf{n}(\psi_0, \theta_0) \]
(or \( \mathbf{r}(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) = p \))

- \( (\pm) \) the principal direction corresponding to zero curvature radius on the coordinate surface.

- \( (\pm) \) the principal direction corresponding to zero curvature radius on the coordinate surface.
say. If the radius of curvature of \( \gamma \) is \( \mu \), then (assuming continuity of the derivative) it follows that

\[
(R_1|R_{1\psi}|/h_1)(\psi_0, \theta_0) = \mu(p).
\]

(2.1.17)

This result can be extrapolated. If \( \bar{R}_1(\psi, \theta, \sigma_0) \) is the principal radius of curvature of the coordinate surface

\[
r(\psi, \theta, \sigma_0) = S(\psi, \theta) + \sigma_0 n(\psi, \theta), \quad \sigma_0 < R_1(\psi, \theta)
\]
corresponding to the principal direction \( S_\psi \), then (1.1.9)

\[
\bar{R}_1(\psi, \theta, \sigma_0) = R_1(\psi, \theta) - \sigma_0,
\]

while a scaling function for this surface is

\[
|r_\psi(\psi, \theta, \sigma_0)| = \bar{R}_1(\psi, \theta, \sigma_0)
\]

\[
= ((R_1 - \sigma_0)h_1/R_1)(\psi, \theta).
\]

(2.1.18)

It follows from these results that

\[
(R_1|R_{1\psi}|/\bar{R}_1)(\psi_0, \theta_0, \sigma_0) = \mu(p)
\]

(2.1.19)
or

\[
(R_1|\nabla \bar{R}_1 \cdot S_\psi|/h_1)(\psi_0, \theta_0, \sigma_0) = \mu(p).
\]

(2.1.20)

The result is still true in the limit as \( \sigma_0 \to R_1(\psi_0, \theta_0) \) and \( \bar{R}_1 \to 0 \), where it is convenient to identify a unit vector normal to the caustic surface as \( \nu(\psi_0, \theta_0) = (S_\psi/h_1)(\psi_0, \theta_0) \) and so

\[
\lim_{\sigma_0 \to R_1(\psi_0, \theta_0)} |\bar{R}_1 \nu \cdot \nabla \bar{R}_1| = \mu(p)
\]

(2.1.21)

These results will be used in analysing the constant-N solutions of the Helmholtz problem near caustic surfaces; analogous results which apply when \( N \) is not constant will be developed in section 4.3.

### 2.2

As caustic surface formation is usual in geometric optics, a description of the configuration of such surfaces in a non-trivial, constant-N case is appropriate. Their geometry, even when \( N \) is constant, can be very complicated, but the example given below will be reasonably accessible.

While a description of the caustic surfaces associated with a branch of the elliptic hyperboloid

\[
z = (1 + ax^2 + by^2)^{1/2}
\]

\[
\equiv F(x, y), \quad 0 < a < b < 1, \quad 0 < b - a < 1,
\]

(2.2.1)
as initial surface is the ultimate objective, first consideration must be the calculation of principal coordinate lines on it. These will be singular perturbations of the principal coordinate lines on the corresponding branch of the hyperboloid of revolution \((a = b)\), the longitudinal family of which are its intersections with members of the family of coaxial planes

\[
y/x = \beta, \quad 0 < \beta < \infty,
\]

and the other ones are its intersections with the family of parallel planes

\[
z = \Gamma, \quad \Gamma > 1.
\]

More generally, the equation governing the projections \((x(s), y(s), 0)\) of the principal lines on a surface \(z = f(x, y)\) are obtained by solving (for \(dy/dx\)) the quadratic expression

\[
(f_x f_y f_{yy} - f_{xy}(1 + f_y^2))(dy/ds)^2 + (f_{yy}(1 + f_x^2) - f_{xx}(1 + f_y^2)) \frac{dy}{ds} \frac{dx}{ds} = 0.
\]

The projections are then found by integration. Singular points \(U = (x, y)\) of the implied differential equation (2.2.4) occur when the coefficients vanish simultaneously, that is when both

\[
f_{yy}(1 + f_x^2) = f_{xx}(1 + f_y^2)
\]

and

\[
f_{xy} = \frac{f_x f_y f_{yy}}{(1 + f_y^2)} = \frac{f_x f_y f_{xx}}{(1 + f_x^2)}
\]

at some \((x, y)\), and it is noted that a point \(u = (U, f(U))\) on the surface \(z = f\) is one at which its two principal curvature radii are equal, or an umbilic point. Thus for an elliptic paraboloid

\[
z = 1 + (ax^2 + by^2)/2 \equiv g(x, y)
\]

the singular points are

\[
U_{Fk} = \left(0, \pm((b-a)/a)^{1/2}/b\right)
\]

while for the hyperboloid \(z = F\) they are

\[
U_{Hk} = \left(0, \pm((b-a)/(ab(b+1)))^{1/2}\right).
\]

There seems to be no closed form integral of the differential equations obtained from the expression (2.2.4) when \(f = F\), but when \(f = g\) the two integrals
yield families of projections of principal lines

\[
\left( ax^2 + \left( b^{1/2}y + ((b-a)/(ab))^{1/2} \right)^2 \right)^{1/2} \\
\pm \left( ax^2 + \left( b^{1/2}y - ((b-a)/(ab))^{1/2} \right)^2 \right)^{1/2} \\
= m_\pm,
\]

where \( m_+ \) and \( m_- \) are constants. These families might be described as confocal pseudo-ellipses (+ sign) and confocal pseudo-hyperbolae (- sign) with foci at \( U_{\pm} \).

For a surface 

\[
z = (ax^2 + by^2)^{1/2} \equiv G(x, y)
\]

- that is, the elliptic cone which is the 'large \((x^2 + y^2)^{1/2}\)' asymptote of \( F \) - the projections of one family of principal lines is the family of similar ellipses

\[
(a + 1)x^2 + (b + 1)y^2 = n_+
\]

and the other family of projections is the family of radial lines

\[
y/x = n_-, 
\]

where \( n_+ \) and \( n_- \) are constants.

Since \( g \) is the 'small \((x^2 + y^2)^{1/2}\)' approximation to \( F \), and \( G \) is the 'large \((x^2 + y^2)^{1/2}\)' one, it is reasonable to conclude that one family of principal lines on the surface \( z = F \) have projections which evolve from something like confocal pseudo-ellipses (with foci at \( U_{\pm} \)) for \((x^2 + y^2)^{1/2}\) not large into a family approximated by similar ellipses when \((x^2 + y^2)^{1/2}\) is large. The other family of projections are something like confocal pseudo-hyperbolae in the small, with straight radial asymptotes. Thus it is inferred that the surface \( z = F \) has a family of closed principal lines, say \( \theta = \) constant, and a family of open ones, say \( \psi = \) constant.

2.3

Although the next step should be a qualitative description of the caustic surfaces

\[
r_j = S + R_jn
\]

which are induced by the initial surface \( z = F \), it may be helpful to start with some simpler configurations.

If \( a = b \), so that the initial surface (2.2.1) is one of revolution, the caustic surface corresponding to the longitudinal principal lines on the initial surface is the horn-shaped surface of revolution obtained by rotating the plane curve

\[
z(x) = b^{-1}(b + 1)\left(1 + b^{1/3}(x/(b + 1))^{2/3}\right)^{3/2}
\]
about the $z$-axis. The other caustic surface is degenerate - it is the line segment 

$$z \geq (b+1)/b, \quad x = y = 0$$  \hspace{1cm} (2.3.3) 

corresponding to principal lines which are parallels of latitude on the initial surface.

Also recalled is the evolute (a plane curve's caustic) of the ellipse of small eccentricity

$$ax^2 + by^2 = 1, \quad 0 < a < b < 1 \quad \text{and} \quad 0 < (b-a) \ll 1.$$  \hspace{1cm} (2.3.4) 

This evolute is entirely within the ellipse, and roughly astroidal in shape, with cusps on each coordinate axis connected by arcs whose curvature is opposite to that of the adjacent arc of the ellipse. The cusps are associated with local extreme values of the ellipse's radius of curvature function; cusps associated with minima are adjacent to the corresponding arc of the ellipse, while those associated with maxima are on the arc of the ellipse opposite the cusp.

Recall the result of the previous section for \( z = F \). There are two families of principal coordinate lines - $\theta =$ constant, members of which are closed, and $\psi =$ constant, which are open; $r_1, h_1$ and $R_1$ are associated with the former, and $r_2, h_2$ and $R_2$ with the latter.

Tangent directions on $r_1$ say, are given by (see 2.1.15)

$$r_{1\psi} = R_{1\psi} \mathbf{n}$$

at any point, and (2.1.16)

$$r_{1\theta} = S_\theta (1 - R_1/R_2) + R_{1\theta} \mathbf{n}.$$ 

Thus the condition

$$R_{1\psi} = 0$$  \hspace{1cm} (2.3.5) 

will locate singular curves on the surface $r_1$. Since the coordinate lines $\theta =$ constant are closed, symmetry considerations require that these loci (2.3.5) are on the intersection of the plane $y = 0$ with the caustic surface, and on the intersection of the part of the plane $z = 0$ with that surface, excluding the strip (2.2.9)

$$-\left(\frac{(b-a)}{(ab/(b+1))}\right)^{1/2} \leq y \leq \left(\frac{(b-a)}{(ab/(b+1))}\right)^{1/2}.$$  

Similarly, the condition locating singular (plane) curves on the caustic surface $r_2$ is

$$R_{2\theta} = 0;$$  \hspace{1cm} (2.3.6) 

it is satisfied on the initial surface's intersection with the plane $x = 0$, but only for the strip (2.2.9)

$$-\left(\frac{(b-a)}{(ab/(b+1))}\right)^{1/2} \leq y \leq \left(\frac{(b-a)}{(ab/(b+1))}\right)^{1/2},$$
and the singular lines on the caustic surface $r_2$ are on its intersection with this strip. These singular lines will be called *chines* below.

The finite chine on $r_2$ has a cusp at $(x = 0, y = 0, z = 1 + b^{-1})$ since, by symmetry
\[ R_{2\phi} = 0 \] (2.7)
there, and similarly the chine on the intersection of the plane $y = 0$ with $r_1$ has a cusp at $(x = 0, y = 0, z = 1 + b^{-1})$ since
\[ R_{1\theta} = 0 \] (2.8)
at that point.

At points on the initial surface $z = F$ for which $(x^2 + y^2) \ll 1$ the principal curvature radii are
\[ R_1 \approx b^{-1} \]
\[ R_2 \approx a^{-1} \] (2.9)
and since $b > a$ the caustic surface $r_1$ is locally between the initial surface and the caustic surface $r_2$. When $(x^2 + y^2) \gg 1$, and so $R_2 \gg R_1$, the caustic surface $r_2$ has the caustic surface $r_1$ in its interior, thus reversing the situation. (Recall 2.3.3 that when $b = a$, the caustic surface $r_1$ degenerates into the line segment $x = y = 0, z \geq 1 + b^{-1}$, so that for $(b - a)$ small, as assumed, the caustic surface $r_1$ will not be too far from this line segment.) Hence the two caustic surfaces have a closed curve of intersection, two points of which are (2.2.9)
\[ r_\pm = (U_{H\pm}, F(U_{H\pm})) \]
that is, the umbilic points of the initial surface where $R_1 = R_2$, and there is a finite volume bounded by parts of both caustic surfaces.

### 3 The Theory for Constant $N \equiv 1$

#### 3.1

So that the integrals which occur in this theory should more generally converge, artificial damping is introduced in the solution of the wave equation
\[ \nabla^2 H - \epsilon^2 H_{tt} = 0 \] (3.1.1)
whose point solutions are taken to be
\[ H(r, t) = H_1(r) \exp ((i - \delta)t), \quad 0 < \delta \ll 1, \quad t > 0, \] (3.1.2)
which requires the spatial factor
\[ H_1(r) = |r|^{-1} \exp ((i - \delta)|r|/\epsilon). \] (3.1.3)
The solution to some boundary value problem for the reduced wave equation

\[ H_1(r, \epsilon, \delta) \]

\[ = (2\pi\epsilon)^{-1} \iint_{S} \frac{K(\psi, \theta) \exp \left( \frac{(1 - \delta)|r - S(\psi, \theta)|}{\epsilon} \right) h_1(\psi, \theta) h_2(\psi, \theta)}{|r - S(\psi, \theta)|} \, d\psi \, d\theta \]  

(3.1.4)

will be evaluated in principle with the ratio \( \delta/\epsilon \ll 1 \), but because the factor \( \exp(-\delta|\mathbf{r} - \mathbf{S}(\psi, \theta)|/\epsilon) \) has only a slight influence on the subsequent explicit approximations - at least for \( \min|\mathbf{r} - \mathbf{S}| \) not large - it will be omitted. Nevertheless it should be recognized that without such a factor the integral

\[ H_1(r, \epsilon) \]

\[ = (2\pi\epsilon)^{-1} \iint_{S} \frac{K(\psi, \theta) \exp \left( \frac{1}{\epsilon} |r - S(\psi, \theta)| \right) h_1(\psi, \theta) h_2(\psi, \theta)}{|r - S(\psi, \theta)|} \, d\psi \, d\theta \]  

(3.1.5)

will not converge unless a suitable decay rate is imposed on the factor \( K(\psi, \theta) \) for values of \((\psi, \theta)\) corresponding to \(|\mathbf{S}(\psi, \theta)| \to \infty\). The factor \( K \) is otherwise supposed to be real, uniformly bounded and analytic everywhere.

The approximate evaluation of the integral (3.1.4) is, in the first instance, by the method of stationary phase for multiple integrals [6]. Thus the stationary points of the integrand’s exponential function’s argument are located by solving simultaneously the conditions

\[ |\mathbf{r} - \mathbf{S}(\psi, \theta)|_\psi = \left( \frac{\mathbf{S}_\psi \cdot (\mathbf{S} - \mathbf{r})}{|\mathbf{S} - \mathbf{r}|} \right)(\psi, \theta) = 0 \]  

(3.1.6)

and

\[ |\mathbf{r} - \mathbf{S}(\psi, \theta)|_\theta = \left( \frac{\mathbf{S}_\theta \cdot (\mathbf{S} - \mathbf{r})}{|\mathbf{S} - \mathbf{r}|} \right)(\psi, \theta) = 0 \]  

(3.1.7)

with \( \mathbf{r} \) given. The solution of these two conditions (3.1.6, 3.1.7) is, in the notation of section 2

\[ \mathbf{r} - \mathbf{S}(\psi, \theta) = \sigma \mathbf{n}(\psi, \theta) \]  

(3.1.8)

- that is, all \((\psi_k, \theta_k)\) pairs such that the associated geometric optics ray

\[ \mathbf{S}(\psi_k, \theta_k) + \sigma \mathbf{n}(\psi_k, \theta_k) = \mathbf{r} \]  

(3.1.9)

for some \( \sigma = \sigma_k \), or what is the same thing, \((\psi_k, \theta_k)\) pairs such that the associated normal to the initial surface passes through the field point.

Given that the principal curvature radii have the same sign, the sense of \( \mathbf{n} \) makes \( \sigma \) positive for field points \( \mathbf{r} \) on the concave side of the initial surface. (This is the simplest case; the subsequent arguments can be modified if \( R_1 \) and \( R_2 \) have opposite signs, but they must be non-zero.) Assume that all the roots of (3.1.6) and (3.1.7) \((\psi_k, \theta_k), k = 1, \ldots, N \) are isolated; some cases of coincident roots will be treated later.
The stationary phase method requires construction of the local of the local quadratic approximation to \(|r - S|\) in the neighbourhood of each root \((\psi_k, \theta_k)\). For example, since

\[ |r - S|_{\theta}(\psi_k, \theta_k, \sigma_k) = 0 \]

at the stationary point for the given field point \(r\), then on omitting its vanishing terms the second derivative

\[
\frac{|r - S|_{\theta\theta}(\psi_k, \theta_k, \sigma_k)}{|r - S|} = \left( \frac{S_{\theta\theta} \cdot (S - r) + S_{\theta} \cdot S_{\theta}}{|r - S|} \right)(\psi_k, \theta_k, \sigma_k)
\]

say, is directly expressible in terms of the principal curvature radius \(R_2\), the scaling function \(h_2\) and the coordinate \(\sigma\) which locates the field point. The coefficient \(M\) is positive for field points on the concave side of \(S\) with \(\sigma < R_2\), and negative for \(\sigma > R_2\); it is always positive on the negative side of \(S\) (where \(\sigma < 0\)). It follows similarly that the coefficient

\[
N(\psi_k, \theta_k, \sigma_k) \equiv \frac{|r - S|_{\psi\psi}}{\sigma} = \left( \frac{h_1^2 (R_1 - \sigma)}{|R_1|} \right)(\psi_k, \theta_k, \sigma_k), \quad \sigma \neq 0. \tag{3.1.11}
\]

Furthermore, because the coordinates \((\psi, \theta)\) are principal on \(S\), the mixed partial derivative is
\[
|r - S|_\psi (\psi_k, \theta_k, \sigma_k) \\
= \left( - S_{\psi} \cdot n + \frac{S_{\theta} \cdot S_{n}}{|r - S|} \right)(\psi_k, \theta_k, \sigma_k) \\
= 0,
\]
(3.1.12)

since each of the constituent terms in its evaluation vanishes separately.

The contribution to the integral (3.1.4) from the neighbourhood of the stationary point \((\psi_k, \theta_k)\) is approximated asymptotically \((\epsilon \to 0, \ |r - S|/(\psi_k, \theta_k) \text{ not small})\) as

\[
(2\pi c)^{-1} \left( \frac{K h_1 h_2 \exp(|r - S|/c)}{|r - S|} \right) (\psi_k, \theta_k, \sigma_k) \\
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( \frac{i}{2c} \left( M(\psi_k, \theta_k, \sigma_k)(\theta - \theta_k)^2 + N(\psi_k, \theta_k, \sigma_k)(\psi - \psi_k)^2 \right) \right) \, d\psi \, d\theta \\
= \left( \frac{K h_1 h_2 |MN|^{-1/2}}{|\sigma|} \right) (\psi_k, \theta_k, \sigma_k) \\
\times \exp \left( i \left( |\sigma|/c -(\pi/4)(2 - \text{sgn}(R_1 - \sigma) - \text{sgn}(R_2 - \sigma)) \right) \right) (\psi_k, \theta_k, \sigma_k) \\
= \left( \frac{K (R_1 R_2)^{1/2}}{|(R_1 - \sigma)(R_2 - \sigma)|^{1/2}} \right) (\psi_k, \theta_k, \sigma_k) \\
\times \exp \left( i \left( |\sigma|/c -(\pi/4)(2 - \text{sgn}(R_1 - \sigma) - \text{sgn}(R_2 - \sigma)) \right) \right) (\psi_k, \theta_k, \sigma_k).
\]
(3.1.13)

Both real and imaginary parts of the expression (3.1.13) exist and are continuous in the limit \(\sigma \to 0\); the limit is \((K(\psi_k, \theta_k) + \eta))\). It is not too difficult to show that this is also the limit of the integral (3.1.4) as \(r \to S\) along the normal \(n(\psi_k, \theta_k)\). The real part of (3.1.13) - depending as it does on \(\cos(|\sigma|/c)\) - is thus infinitely differentiable in \(\sigma\) at \(\sigma = 0\), while the imaginary part, depending on \(\sin(|\sigma|/c)\), is continuous but has a discontinuous derivative. So the real part of (3.1.13) describes a displacement initial condition, and the imaginary part an impulsive initial condition.

So far the contribution to the integral (3.1.4) from a single stationary point \(k^{th}\) has been calculated. To approximate the integral, the contributions from all stationary points must be summed. If the field point can be located

\[
r = S(\psi_k, \theta_k) + \sigma_k n(\psi_k, \theta_k), \quad k = 1, \ldots, N
\]
the formal result is

\[ H_1(r, \epsilon, 0) \approx \sum_{i} \left( \frac{K(R_1R_2)^{1/2}}{|(R_1 - \sigma_k)(R_2 - \sigma_k)|^{1/2}} \right) \times \exp \left( i \left( |\sigma_k| \sqrt{\epsilon} - (\pi/4)(2 - \text{sgn}(R_1 - \sigma_k) - \text{sgn}(R_2 - \sigma_k)) \right) \right) (\psi_k, \theta_k). \]

(3.1.14)

When the field point is on the convex side of \( S \), or between it and the nearest caustic surface, there is only one geometric optic ray (or normal to \( S \)) through it, so \( N = 1 \) in (3.1.14). For the elliptic-hyperbolic initial surface discussed in section 2, in the regions between the two caustic surfaces there are three rays through any given field point, and in the region interior to both caustic surfaces (i.e. near the \( z \)-axis, but \( z > (1 + a^{-1}) \)) there are five such rays, so \( N = 3 \) and \( N = 5 \) respectively.

The formula (3.1.14) is well known; the distinction between the current theory and the more usual ones (see, for example [4], [7]) is that it is obtained without recourse to local solution(s) near the caustic surface to provide connection formulae for geometric optic solutions on non-singular ray segments.

### 3.2

In this subsection an amalgamation of the method of stationary phase and its extension [1] is used to describe the near caustic region of the solution domain. To simplify calculations when the field point \( r \) is near a caustic surface, but not near a chine, its location is described in a modified way (rather than as in 2.1.8).

Suppose \( r(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) \) is a point of the caustic surface \( r_1 \), then for any \( \psi_\ast \) and \( \alpha \) (but as it turns out here, with both \( |\psi_0 - \psi_\ast| \ll 1 \) and \( 0 < \alpha \ll 1 \)), one can define a unit vector

\[ u(\psi_\ast, \theta_0, \alpha) = n(\psi_\ast, \theta_0) \cos(\alpha) + S_\psi(\psi_\ast, \theta_0) \sin(\alpha)/h_1(\psi_\ast, \theta_0). \]

(3.2.1)

The field point under consideration will now be specified as

\[ r = u(\psi_\ast, \theta_0, \alpha)R_1(\psi_0, \theta_0)/\cos(\alpha) + S(\psi_\ast, \theta_0), \]

(3.2.2)

with \( \psi_\ast \) now chosen so that, for \( r \) specified as in (3.2.2),

\[ |r - S|\varphi(\psi_\ast, \theta_0) = 0, \]

(3.2.3)

or so that

\[ \left( \frac{S_{\psi\psi} \cdot (S - r) + S_{\psi} \cdot S_{\psi}}{|r - S|} - \frac{(S_\psi \cdot (S - r))^2}{|r - S|^3} \right)(\psi_\ast, \theta_0) = \frac{\cos(\alpha)h_1^2(\psi_\ast, \theta_0)}{R_1(\psi_0, \theta_0)} \left( \cos^2(\alpha) - \frac{R_1(\psi_0, \theta_0)}{R_1(\psi_\ast, \theta_0)} \right) - \sin(\alpha)h_1(\psi_\ast, \theta_0) \]

\[ = 0 \]

(3.2.4)
If \( \psi_* = \psi_0 \), then (3.2.4) is satisfied with \( \alpha = 0 \). When \( |\psi_* - \psi_0| \) is small, so must be \( |\alpha| \), and the critical term in (3.2.4) is approximated as

\[
R_1(\psi_*, \theta_0) \approx R_1(\psi_0, \theta_0) + (\psi_* - \psi_0)R_{1,\psi}(\psi_0, \theta_0);
\]

and its derivative can be approximated by their values at \((\psi_0, \theta_0)\), so the approximate solution is

\[
\psi_* - \psi_0 \approx \alpha \left( \frac{h_{1,\psi}R_1^2}{h_1^2R_{1,\psi}} \right)(\psi_0, \theta_0).
\] (3.2.5)

If \( R_{1,\psi}(\psi_0, \theta_0) = 0 \), the field point would be near a chine of the caustic surface (see section 2.3); this singular case will be discussed later.

Take, for example, the case in which both \( h_{1,\psi} > 0 \) and \( R_{1,\psi} > 0 \), so that \( r(\psi_* > \psi_0, \theta_0, R_1(\psi_0, \theta_0)) \) is a point inside the caustic surface. Then the field point \( r \) is found at (3.2.2)

\[
r = S(\psi_0, \theta_0) + R_1(\psi_0, \theta_0) \left( n(\psi_0, \theta_0) + \tan(\alpha)S_\psi(\psi_0, \theta_0)/h_1(\psi_0, \theta_0) \right),
\] (3.2.6)

and since (from 3.2.5) \( |\psi_* - \psi_0| \sim \mathcal{O}(\alpha) \), and \( \alpha \) is small, this is close to the point

\[
r = S(\psi_0, \theta_0) + R_1(\psi_0, \theta_0) \left( n(\psi_0, \theta_0) + \alpha S_\psi(\psi_0, \theta_0)/h_1(\psi_0, \theta_0) \right).
\] (3.2.7)

This in turn is near the point

\[
r = S(\psi_0, \theta_0) + R_1(\psi_0, \theta_0)n(\psi_0, \theta_0)
\]

(3.2.8)

with \( \psi_0 \) chosen so that

\[
\alpha R_1(\psi_0, \theta_0) = \int_{\psi_0}^{\psi_*} |r_\psi(t, \theta_0, R_1(\psi_0, \theta_0))| dt
\]

\[
= \int_{\psi_0}^{\psi_*} h_1(t, \theta_0) \frac{R_1(t, \theta_0) - R_1(\psi_0, \theta_0)}{R_1(\psi_0, \theta_0)} dt
\]

\[
\approx \left( \frac{\psi_0 - \psi_0}{2} \right) \left( \frac{B_{1,\psi}h_1}{R_1} \right)(\psi_0, \theta_0),
\] (3.2.9)

on using (2.1.18) (with \( \sigma_0 = R_1(\psi_0, \theta_0) \)), and the two term Taylor series expansion in \( \psi \) of \( R_1 \) at \( \psi_0 \).

From this last result (3.2.9) it is seen that \( |\psi_0 - \psi_0| \sim \mathcal{O}(\alpha^{1/2}) \), and because \( |\psi_* - \psi_0| \sim \mathcal{O}(\alpha) \) (by 3.2.5), the approximation of \( \psi_* \) by \( \psi_0 \) is justified. (On occasions the distinction may be re-introduced below.)

On recalling from section 2.1 that \( S_\psi(\psi_0, \theta_0) \) is (parallel to the) normal to the caustic surface at \( r_1(\psi_0, \theta_0) \), the field point \( r \) now taken to be described as in (3.2.8), is distant approximately

\[
\mathcal{F} = \alpha R_1(\psi_0, \theta_0)
\] (3.2.10)
from the nearest point on the caustic surface. Of course there are in fact two roots of equation (3.2.9), namely
\[(\psi_a - \psi_0) = \pm \left( R_1 (2\alpha / (R_1 \varphi_1))^{1/2} \right) (\psi_0, \theta_0), \quad (3.2.11)\]
so if the choice is made \((\psi_a - \psi_0) > 0\), the other is
\[2\psi_0 - \psi_a = \psi_0 - (\psi_a - \psi_0). \quad (3.2.12)\]
This root identifies a second ray passing through the field point, that is the ray
\[r = S(2\psi_0 - \psi_a, \theta_0) + \epsilon n(2\psi_0 - \psi_a, \theta_0). \]
Hence the field point (as specified in 3.2.2) is near
\[r \approx r(2\psi_0 - \psi_a, \theta_0, R_1(\psi_0, \theta_0)) \quad (3.2.13)\]
as well as being close to \(r(\psi_a, \theta_0, R_1(\psi_0, \theta_0))\).

By (exactly) locating the field point as in (3.2.2), the calculation of the coefficients in the Taylor expansion of \(|r - S|\) is simplified. The expansion, in view of the condition (3.2.3), is
\[|r - S| = |r - S| = A(\psi - \psi_\ast)^2/6 + B(\psi - \psi_\ast)^2 + M(\theta - \theta_\ast)^2/2 + E(\psi - \psi_\ast)(\theta - \theta_\ast)^2/2 + G(\theta - \theta_\ast)^3/6 + \text{t.h.d}, \quad (3.2.14)\]
where the abbreviation t.h.d. denotes terms of higher degree; the bracketed terms will not affect the ultimate asymptotic result. (There is no \((\theta - \theta_0)\) term in (3.2.14) since
\[l_r$ - S_l(\psi, \theta) = (S - S)(\psi_\ast, \theta_0) = 0\]
as a consequence of 3.2.1 and 3.2.2.)

One coefficient in (3.2.14) is
\[B = |r - S|_\psi(\psi_\ast, \theta_0) \approx -(h_1 F/R_1)(\psi_0, \theta_0), \quad (3.2.15)\]
another is
\[C = |r - S|_\theta(\psi_\ast, \theta_0) \approx -(h_1 F/R_1)(\psi_0, \theta_0), \quad (3.2.16)\]
and the remaining one needed explicitly is
\[A = |r - S|_\varphi(\psi_\ast, \theta_0) \approx (h_1^2 R_1 \varphi_1/R_1^2)(\psi_0, \theta_0) > 0. \quad (3.2.17)\]
(The coefficient $M$ is sufficiently well approximated in (3.1.10); it is here

$$M = |\mathbf{r} - \mathbf{S}|_{\delta \delta}(\psi, \theta_0) \approx M(\psi_0, \theta_0, R_1(\psi_0, \theta_0)).$$

Now change the (ultimately, integration) variables in the expression (3.1.14) through

$$\psi - \psi_* = \epsilon^{1/3}\xi,$$
$$\theta - \theta_0 = \epsilon^{1/2}\eta,$$  \hspace{1cm} (3.2.18)

and rescale the parameter

$$\mathcal{F} = \epsilon^{2/3}\xi,$$  \hspace{1cm} (3.2.19)

so that the Taylor series representation (3.2.14) becomes

$$\left(\mathcal{F} - \mathcal{F}^*\right) \approx \epsilon^{1/3}A{\xi}^3/6 + b\xi + M\eta^2/2 + \epsilon^{1/3}G\eta^2/2 + \epsilon^{1/2}G\eta^2/6 + \text{t.h.d.},$$  \hspace{1cm} (3.2.20)

where

$$b = \epsilon^{-2/3}B \approx -(f_{11}/R_1)(\psi_0, \theta_0)$$  \hspace{1cm} (3.2.21)

and

$$c \approx -(f_{12}/R_1)(\psi_0, \theta_0).$$  \hspace{1cm} (3.2.22)

The next step is to construct a perturbation variable change

$$\xi = u + \xi_1(u, v),$$
$$\eta = v + \eta_1(u, v),$$  \hspace{1cm} (3.2.23)

where $\xi_1$ and $\eta_1$ are polynomials of least degree two, each with a small multiplicative coefficient, and chosen so that the transformed series (3.2.20) is truncated:

$$\left(\mathcal{F} - \mathcal{F}^*\right) \approx \epsilon^{1/3}A{u}^3/6 + bu + Mv^2.$$  \hspace{1cm} (3.2.24)

Because the coefficients $b$ and $c$ can vanish (when $\epsilon=0$), the variable change cannot be done uniformly. Thus when $b$ and $c$ are not zero, it is

$$\xi = u - \epsilon^{1/2}(c/b)uv + \text{t.h.d.},$$
$$\eta = v - \epsilon^{1/2}(E/(2M))uv - \epsilon^{1/2}(G/(6M))v^2 - \epsilon^{1/2}(D/(2M))u^2 + \text{t.h.d.};$$  \hspace{1cm} (3.2.25)

but when $b=c=0$ the first variable change (3.2.25) is replaced with

$$\xi = u - \epsilon^{1/2}(L/(12A))u^2 + \text{t.h.d.}$$  \hspace{1cm} (3.2.26)
where
\[ L = |r - S|_{\psi \psi}(\psi_*, \theta_0). \] (3.2.27)

This non-uniformity is of no present concern, as \( c/b = (h_2/h_1) \) is always finite, the Jacobian determinant
\[ J(u, v) = \det \begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} \]
evaluates in both cases with
\[ J(0, 0) = 1, \] (3.2.28)
and the leading term to the resulting approximations to (3.1.4) turns out to be uniform in \( f \) on any closed interval including \( f = 0 \).

The contribution to the integral (3.1.4) from that part of the range of integration near \((\psi_*, \theta_0)\) for a near caustic field point \( r(\psi_*, \theta_0, R_1(\psi_0, \theta_0)) \) is approximated as
\[
(2\pi \iota)^{-1} \left( \iota^{5/6} J(0, 0) h_1 h_2 |r - S|^{-1} \exp(i|r - S|/\iota) \right)(\psi_*, \theta_0)
\]
\[
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( i(Au^3/6 + bu + Me^2/2) \right) du dv
\]
\[
\simeq (2\pi)^{-1/2} \iota^{-1/6} \left( K h_1 h_1 h_2 \right)^{-1/2} \left( 2 - \text{sgn}(R_2 - \sigma) \right)^{-1/2}
\]
\[
\times \exp \left( i |r - S|/\iota - (\pi/4)(2 - \text{sgn}(R_2 - \sigma)) \right)(\psi_0, \theta_0)
\]
\[
\times \int_{-\infty}^{\infty} \cos(Au^3/6 + bu) du. \] (3.2.29)

The approximations
\[ \sigma = |r - S|(\psi_*, \theta_0) \approx |r - S|(\psi_0, \theta_0) \approx R_1(\psi_0, \theta_0) \] (3.2.30)
have been used selectively in the evaluation (3.2.29); the last integral is standard (see, for example [3])
\[
\int_{-\infty}^{\infty} \cos(Au^3/6 + bu) du = 2\pi(2/A)^{1/3} \text{Ai}\left((2/A)^{1/3} b \right) \] (3.2.31)
where \( \text{Ai}(z) \) is the Airy integral
\[ \text{Ai}(z) = (2\pi)^{-1} \int_{-\infty}^{\infty} \cos(s^3/3 + zs) ds. \] (3.2.32)

Approximate evaluation of the integral (3.1.4) is completed by adding the contributions to it from other rays the field point, as in section 3.2. These contributions will be \( O(1) \) and hence less significant than the contribution (3.2.29).
Then the result (3.2.29) is

\[ (2\pi)^{1/2}e^{-1/6} \left( (2/A)^{1/3}K h_1(R_2/R_1)^{1/2} |R_2 - \sigma|^{-1/2} \right. \]

\[ \times \exp \left( a(|r - S|/\epsilon - (\pi/4)(2 - \text{sgn}(R_2 - \sigma))) \right) \left( \psi_0, \theta_0 \right) \]

\[ \times Ai\left( \left( \frac{2h_1}{R_1R_{1\psi}} \right)^{1/3} \right), \]

(3.2.33)

and it is noted that a combination occurring here has earlier been identified in equations (2.1.17) and (2.1.20)

\[ \left( \frac{R_1R_{1\psi}}{h_1} \right)(\psi_0, \theta_0) = \mu(p) = \lim_{\sigma \to R_1} (R_1 \nu \cdot \nabla R_1)(\psi_0, \theta_0) \]

(3.2.34)

and so in terms of a local geometric parameter \( \mu \) of the caustic surface, or in terms of a local geometric parameter \( R_1 \) of the constant-\( \sigma \) surface where it terminates at the caustic surface.

The approximate evaluation of the integral (3.1.4) when the field point is near regular points of the caustic surface \( R_2 \) can be obtained similarly.

If the field point lies near the intersection of the two caustic surfaces, but not near singularities of either, then it will be near a point

\[ r(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) = r(\psi_0', \theta_0', R_2(\psi_0', \theta_0')) \]

with \( R_{1\psi}(\psi_0, \theta_0) \neq 0 \) and \( R_{2\psi}(\psi_0', \theta_0') \neq 0 \), and with \( S(\psi_0, \theta_0) \) and \( S(\psi_0', \theta_0') \) locating distinct points on the initial surface. In this case the approximation to the integral (3.1.4) will consist of the sum of two terms of the type (3.2.29), one for each of \( (\psi_0, \theta_0) \) and \( (\psi_0', \theta_0') \).

If the points \( S(\psi_0, \theta_0) \) and \( S(\psi_0', \theta_0') \) are coincident, and if \( R_1(\psi_0, \theta_0) = R_2(\psi_0', \theta_0') \) - that is \( (\psi_0 = \psi_0', \theta_0 = \theta_0') \) identifies an umbilical point of \( S \) - the kind of explicit approximation described here cannot be achieved. However intuitive calculations suggest an \( O(\epsilon^{-1/2}) \) order of magnitude result.

The approximation method also fails when the coefficient \( A \) vanishes, that is, from the evaluation (3.2.17), when \( R_{1\psi}(\psi_0, \theta) = 0 \) and the field point is near a chine of the caustic surface as described in section 2.3. This case is discussed in the next section.

The next problem is to see how the approximate evaluation (3.2.33) connects with the results of section 3.1. For large negative values of the parameter \( b \) - that is, field point inside the caustic surface \( r_1 \), but not too far from it \((|b| \propto f = \epsilon^{-3/3}F)\) - the large negative argument asymptotic approximation to (3.2.32) is

\[ Ai(z) \sim \pi^{-1/2}|z|^{-1/4} \cos(2|z|^{3/3} - \pi/4) \quad z \to -\infty \]
[3], so as \( b \to -\infty \) the asymptotic result is

\[
(2/A)^{1/3} \text{Ai}(2(Ab)^{1/3}) \sim \pi^{-1/2} (2/(A|b|))^{1/4} \cos \left( \frac{2(2/A)^{1/2}|b|^{3/2}}{3 - \pi/4} \right).
\]

(3.2.35)

To interpret the terms occurring when this result (3.2.35) is used (through 3.2.31) in (3.2.29), some earlier geometry is again recalled. The plane curve of intersection of the tangent plane to the coordinate surface \( \theta = \theta_0 \) at \( \psi = \psi_0 \) with the caustic surface \( r_1 \) has radius of curvature (2.1.17) \( R_{1\psi} > 0 \) here

\[
F(p) = (R_{1\psi}/b_1)(\psi_0, \theta_0)
\]

at the point \( r(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) = p \). If \( D > 0 \) is the distance from the field point \( r(\psi_a, \theta_0, R_1(\psi_0, \theta_0)) \) to the caustic, but measured along the normal (ray) from \( S(\psi_a, \theta_0) \), then by elementary geometry

\[
D \approx (2\mu F)^{1/2}
\]

so that the amplitude coefficient in (3.2.35) can be expressed as

\[
(2/(A|b|))^{1/4} = \epsilon^{1/6}(2/D)^{1/2}(R_1/h_1)(\psi_0, \theta_0).
\]

(3.2.36)

(3.2.37)

It is also the case that \( D \) is the distance from the caustic to the field point, but measured along the other, nearly coincident normal from \( S(2\psi_0 - \psi_a, \theta_0) \) which also passes through the field point. (This near coincidence distinguishes the specified ray from any others which may also pass through the field point.)

Part of the trigonometric argument in (3.2.35) is

\[
2(2/A)^{1/2}|b|^{3/2}/3 = \epsilon^{-1} \left( (\psi_a - \psi_0)|r - S|_\psi + (\psi_a - \psi_0)^3|r - S|_{\psi(\psi_0/6)} \right)(\psi_0, \theta_0),
\]

(3.2.38)

on using the evaluations of \( A \) and \( b \) (3.2.17 and 3.2.21), and the relations connecting \( (\psi_a - \psi_0) \) and \( F \) (3.2.9 and 3.2.10).

When these identifications (3.2.37 and 3.2.38) are substituted in the asymptotic result (3.2.35) and this in turn substituted in (3.2.33) then the asymptotic form (in the sense 'away from caustic') of the near caustic result follows. To shorten the particular case \( R_1 < R_2 \) (and \( \text{sgn}(R_2 - \sigma) = 1 \)) in consequence) is taken. Then, for example the real part of the asymptotic form of (3.2.33) is

\[
K \left( R_1 R_2 / (D(R_2 - \sigma)) \right)^{1/2} \times \cos \left( \epsilon^{-1} \left( |r - S| + (\psi_a - \psi_0)|r - S|_\psi + (\psi_a - \psi_0)^3|r - S|_{\psi(\psi_0/6)} \right) \right)
\]

\[+
K \left( R_1 R_2 / (D(R_2 - \sigma)) \right)^{1/2} \times \sin \left( \epsilon^{-1} \left( |r - S| - (\psi_a - \psi_0)|r - S|_\psi - (\psi_a - \psi_0)^3|r - S|_{\psi(\psi_0/6)} \right) \right)
\]

(3.2.39)
with evaluations of $K$, $R_1$, $R_2$ and $|r - S|$, and derivatives, at $(\psi_0, \theta_0)$ and with $\sigma = R_1(\psi_0, \theta_0)$.

The arguments of the trigonometric terms in (3.2.39) are the truncated Taylor expansions in $\psi$ about $\psi_0$ of $|r - S|\psi\psi_0(\psi_0, \theta_0)$, when evaluated at $\psi = \psi_0$ and $\psi = 2\psi_0 - \psi_a$ respectively, neglecting as negligible the second derivative term whose coefficient $|r - S|\psi\psi_0(\psi_0, \theta_0, R_1(\psi_0, \theta_0))$ is very small ($\psi_0 \approx \psi_a$, and

$$|r - S|\psi\psi_0(\psi_0, \theta_0, R_1(\psi_0, \theta_0)) = 0$$

The two parts of (3.2.39) can now be interpreted separately. If $R_1$, $R_2$, $K$ and $S$ do not vary too rapidly near $(\psi_0, \theta_0)$, then both approximations

$$\left( K \left( R_1R_2/(R_2 - \sigma) \right)^{1/2} \right)(\psi_0, \theta_0) \approx \left( K \left( R_1R_2/(R_2 - \sigma) \right)^{1/2} \right)(\psi_a, \theta_0)$$

$$\approx \left( K \left( R_1R_2/(R_2 - \sigma) \right)^{1/2} \right)(2\psi_0 - \psi_a, \theta_0)$$

are acceptable, and the expression (3.2.39) can be seen as the sum of two geometrical optics solutions on rays from points $S(\psi_a, \theta_0)$ and $S(2\psi_0 - \psi_a, \theta_0)$ in $S$, provided $\sigma$ and $D$ are appropriately interpreted for each ray:

$$\sigma = |r - S|(\psi_a, \theta_0, R_1(\psi_0, \theta_0))$$

or

$$\sigma = |r - S|(2\psi_0 - \psi_a, \theta_0, R_1(\psi_0, \theta_0))$$

and

$$D = \left( R_1(\psi_a, \theta_0) - |r - S|(\psi_a, \theta_0, R_1(\psi_0, \theta_0)) \right)$$

or

$$D = \left( |r - S|(2\psi_0 - \psi_a, \theta_0, R_1(\psi_0, \theta_0)) - R_1(2\psi_0 - \psi_a, \theta_0) \right).$$

(Bear in mind here that, on account of 3.2.13

$$|r - S|(\psi_a, \theta_0, R_1(\psi_0, \theta_0)) = |r - S|(2\psi_0 - \psi_a, \theta_0, R_1(\psi_0, \theta_0)).$$

So the first part of the expression (3.2.29) describes a ray just before it touches its caustic surface, and the second describes the ray just after it touches, and the description is consistent with equation (3.1.13).

When $b$ is positive in (3.2.29), the field point is outside the caustic surface, $\alpha$ is negative in (3.2.9) and that equation has no (real) solution. Thus there is no adjacent ray through the field point. The contribution to the integral is still approximated by (3.2.29), but the Airy integral (3.2.30) decays exponentially for large positive $x$:

$$2\text{Ai}(z) \sim \pi^{-1/2}z^{-1/4}\exp(-2z^{3/2}/3) \quad z \to \infty$$
[3] and so also does the approximation (3.2.29) outside the caustic surface. (The full approximation to (3.1.4) would then contain contributions of the type (3.1.14) from regions near pairs \((\psi, \theta)\) satisfying (3.1.6) and (3.1.7) for the given field point. There will always be at least one such contribution.)

### 3.3

Suppose \((\psi_c, \theta_c)\) identifies a ray through a chine on the caustic surface \(r_1\), and the ray is the critical one

\[
R_{1\psi}(\psi_c, \theta_c) = 0 \tag{3.3.1}
\]

but not from an umbilical point of the initial surface: to be specific, let

\[
R_1(\psi_c, \theta_c) < R_2(\psi_c, \theta_c).
\]

Then the contribution to the integral (3.1.4) arising from the integration domain neighbourhood of \((\psi_c, \theta_c)\) is to be calculated when the field point is

\[
r = r(\psi_c, \theta_c, \sigma)
\]

and near the caustic surface \(r_1\) so that

\[
|\sigma - R_1(\psi_c, \theta_c)| \ll 1. \tag{3.3.3}
\]

As usual, the Taylor coefficients for the expansion of \(|r - S|\) about \((\psi_c, \theta_c)\) with fixed \(r\) are required. Since \(r\) is on the ray, then

\[
|r - S|_\psi = |r - S|_\theta = |r - S|_\theta = 0
\]

at \((\psi_c, \theta_c)\), as in section 3.1. Likewise from that section, and with \(\text{sgn} \sigma = 1\), at \((\psi_c, \theta_c)\) one has

\[
|r - S|_\theta = \left( h_2^2(R_2 - \sigma)/(R_2 \sigma) \right)(\psi_c, \theta_c) \equiv M(\psi_c, \theta_c, \sigma) > 0 \tag{3.3.4}
\]

and

\[
|r - S|_\psi = \left( h_1^2(R_1 - \sigma)/(R_1 \sigma) \right)(\psi_c, \theta_c) \equiv N(\psi_c, \theta_c, \sigma) > 0, \tag{3.3.5}
\]

the latter being small and changing sign as \(\sigma\) passes through \(R_1\). Further, since the field point is on the ray, and \(R_{1\psi} = 0\) on it, then

\[
A = |r - S|_{\psi\psi\psi}(\psi_c, \theta_c) = (3h_1h_1\psi(R_1 - \sigma)/(R_1 \sigma))(\psi_c, \theta_c), \tag{3.3.6}
\]

now, and

\[
G = |r - S|_{\psi\psi\psi}(\psi_c, \theta_c) = (3h_1h_1\psi(R_1 - \sigma)/(R_1 \sigma))(\psi_c, \theta_c) + O(R_1 - \sigma), \tag{3.3.7}
\]

and

\[
D = |r - S|_{\psi\theta}(\psi_c, \theta_c) = \left( h_2^2R_2/(R_1) \right)(\psi_c, \theta_c) + O(R_1 - \sigma), \tag{3.3.8}
\]
with $R_{1\psi\theta}(\psi_c, \theta_c)$ now assumed to be non-small.

The Taylor expansion of $|r - S|$ about $(\psi_c, \theta_c)$ can be expressed as

$$
|r - S|(\psi, \theta) - |r - S|(\psi_c, \theta_c) = \frac{M(\theta - \theta_c)^2}{2} + \frac{N(\psi - \psi_c)^2}{2} + \frac{G(\psi - \psi_c)^4}{24} + \frac{D(\psi - \psi_c)^2(\theta - \theta_c)}{2} + \left(\frac{A(\psi - \psi_c)^3}{6} + \frac{E(\psi - \psi_c)(\theta - \theta_c)^2}{2} + \frac{F(\theta - \theta_c)^3}{6} + \text{t.h.d.}\right)
$$

where the bracketed terms do not rescale the variables in (3.3.9) and affect the ultimate approximation. Now rescale the variables in (3.3.9)

$$
\psi - \psi_c = \epsilon^{1/4}\xi
$$

$$
\theta - \theta_c = \epsilon^{1/2}\eta
$$

and define the parameter

$$
R_1 - \sigma = \epsilon^{1/2}g.
$$

(3.3.11)

(This last definition means that if $R_{1\psi\theta}(\psi_c, \theta_c) > 0$ and $g$ are the same sign, the field point is outside $r_1$.)

With these rescalings and definition (3.3.10, 3.3.11), the Taylor series (3.3.9) is rewritten as

$$
(|r - S|(\psi, \theta) - |r - S|(\psi_c, \theta_c))/\epsilon = \frac{M\eta^2}{2} + \frac{u\xi^2}{2} + \frac{G\xi^4}{24} + \frac{D\xi^2\eta}{2} + \left(\epsilon^{1/4}\Lambda\xi^3/6 + \epsilon^{1/4}\Xi\eta^3/2 + \epsilon^{1/2}F\eta^3/6 + \text{t.h.d.}\right)
$$

(3.3.12)

after making obvious definitions of the coefficients $a$ and $n$. Next, a variable change

$$
\xi = \xi_1(u, v)
$$

$$
\eta = \eta_1(u, v),
$$

where $\xi_1$ and $\eta_1$ are polynomials degree at least two, each with a small coefficient, reduces the series (3.3.12) to a finite polynomial

$$
(|r - S|(\psi, \theta) - |r - S|(\psi_c, \theta_c))/\epsilon = \frac{Mv^2}{2} + \frac{nu^2}{2} + \frac{Gu^3}{24} + \frac{Du^2v}{2},
$$

(3.3.13)

if $g \neq 0$; a different change of the same general form will produce the same outcome when $g = 0$. In both cases the Jacobian determinants of the transformations satisfy

$$
J(0, 0) = \det \left( \begin{array}{cc} \xi_u & \xi_v \\ \eta_u & \eta_v \end{array} \right)(0, 0) = 1,
$$

(3.3.14)

and the approximations to the integral which follow are uniform in $g$. 

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Yet another variable change is required in order that the essential integral in the approximation to (3.1.4) should be separable. This change is

\[ u = U, \quad v = V - DU^2/(2M), \]  

and its Jacobian determinant is

\[ j(U, V) = \det \begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix} (U, V) \equiv 1, \]  

while the polynomial (3.1.13) becomes

\[ (|r - S|(\psi, \theta) - |r - S|(\psi_e, \theta_e))/\epsilon = MV^2/2 + nU^2/2 + GU^4/24 \]  

where

\[ G = G - 3D^2/(2M). \]  

All the foregoing means that the integral (3.1.4) can be written as an approximation

\[ \int_{S} \int (2\pi i)^{-1} \left( \exp(i|r - S|/\epsilon)(\psi_e, \theta_e) \right) \epsilon^{3/4} \]  

\[ \times \int_{S} J_1(U, V) \left( \frac{Kh_1h_2}{|r - S|} \right) (\psi(U, V), \theta(U, V)) \]  

\[ \times \exp \left( i(MV^2/2 + nU^2/2 + GU^4/24) \right) dU dV \]  

\[ \approx (2\pi i)^{-1} \epsilon^{-1/4} \left( \exp(i|r - S|/\epsilon)(\psi_e, \theta_e) \right) J_1(0, 0) \]  

\[ \times \int_{S} \int \exp \left( i(MV^2/2 + nU^2/2 + GU^4/24) \right) dU dV \]  

where the Jacobian determinant is

\[ \epsilon^{3/4} J_1(U, V) = \det \begin{pmatrix} \theta_U & \theta_V \\ \psi_U & \psi_V \end{pmatrix} = \epsilon^{3/4} J(u, v) j(U, V) \]  

and

\[ J_1(0, 0) = 1. \]  

The last integral separates

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( i(MV^2/2 + nU^2/2 + GU^4/24) \right) dU dV = (2\pi/M)^{1/2} \exp(i\text{sgn}(M)/4) \int_{-\infty}^{\infty} \exp \left( i(nU^2/2 + GU^4/24) \right) dU, \]  

(3.3.21)
and a closed (if complicated) form evaluation of the remaining integral can be identified using the formulae stated in [2]

\[ \begin{align*}
  p.23 & \quad \text{formula 4} \\
p.24 & \quad \text{formula 9} \\
p.83 & \quad \text{formulae 2, 4.}
\end{align*} \]

A similar analysis will furnish the approximation to the integral (3.1.4) when the field point is on a critical (i.e., with \( R_{2\theta} = 0 \)) ray passing through a chine on the other caustic surface \( r_2 \), and near it.

The analysis of this section will fail if \( M \) and \( N \) vanish simultaneously at a point on a given ray - that is, if \( (\psi_C, \theta_C) \) identifies a ray from an umbilic of the initial surface, and

\[ \sigma = R_1(\psi_C, \theta_C) = R_2(\psi_C, \theta_C). \]

In this case the Taylor series representation of

\[ |r - S(\psi, \theta) - |r - S(\psi_C, \theta_C) \]

in perturbation variables does not generally lead to a separable double integral in the approximation process. However an order of magnitude estimate

\[ |H| = \mathcal{O}(\epsilon^{-1/2}) \]

can be made for such field points, as noted earlier.

### 4 Geometrical Optics

#### 4.1

The material in the following subsection is a summary of that in [4].

If a formal solution of equation (1.1.1) is the series

\[ H = A \cos(\phi / \epsilon) + \sum_{n=1}^{\infty} \epsilon^n H_n, \quad (4.1.1) \]

its leading term is the geometrical optics approximation to \( H \). Standard techniques of applied mathematics operating on equation (1.1.1) using the ansatz \((4.1.1)\) produce the characteristic ray equation, in canonical form

\[ \frac{d^2r}{dt^2} = (1/2) \nabla(N^2) \equiv U(r), \quad (4.1.2) \]

where a vector field \( U \), whose components with respect to \((x_1, x_2, x_3)\) Cartesian axes are \((U_1, U_2, U_3)\), is introduced for convenience. On any given ray the (scaled) phase \( \phi \) satisfies

\[ \frac{d\phi}{dt} = N^2(r(t)) \quad (4.1.3) \]
and the arc length \( \sigma \) along the ray satisfies
\[
\frac{d\sigma}{dt} = N(r(t)) = |\frac{d}{dt}r| > 0. \tag{4.1.4}
\]

Initial conditions for these equations are
\[
\begin{align*}
(r(0; \psi, \theta)) & = S(\psi, \theta), & \frac{dr}{dt}(0, \psi, \theta) = (nN)(S(\psi, \theta)) \tag{4.1.5} \\
\phi(0; \psi, \theta) & = 0, \tag{4.1.6} \\
\text{and} \quad \sigma(0; \psi, \theta) & = 0 \tag{4.1.7}
\end{align*}
\]
respectively, where \( \psi \) and \( \theta \) are again principal coordinates in the initial surface \( S \), and \( n \) is its normal vector. A solution of (4.1.2) satisfying (4.1.5) is a ray, and denote any particular ray solution as
\[
r(t; \psi, \theta) \equiv \tilde{r}(t) \tag{4.1.8}
\]
for brevity.

The characteristic equation governing the amplitude \( A \) integrates in terms of a new unknown \( \Delta(t; \psi, \theta) = \Delta(\tilde{r}(t)) \)
\[
A(\tilde{r}(t)) = A(\tilde{r}(0)) \left( N(\tilde{r}(0))\Delta(\tilde{r}(0))/\left[N(\tilde{r}(0))\Delta(\tilde{r}(0))\right]\right)^{1/2} \tag{4.1.9}
\]
it turns out that the product \( NA \) can be identified as
\[
(N\Delta)(\tilde{r}(t)) = \left| \frac{\partial \tilde{r}}{\partial \psi} \times \frac{\partial \tilde{r}}{\partial \theta} \cdot \frac{d}{dt}(\tilde{r}(t)) \right| \tag{4.1.10}
\]
on a given ray. So in particular, one has
\[
\Delta(\tilde{r}(0)) = (h_1h_2)(\psi, \theta). \tag{4.1.11}
\]
(The ratio \( \Delta(\tilde{r}(t))/\Delta(\tilde{r}(0)) \) can be shown to be independent of the choice of smooth coordinization of \( S \).) An indirect calculation of the partial derivatives in (4.1.10) will be used below.

### 4.2
As first steps the integration of the ray equation (4.1.2) and its perturbation, for given \( \tilde{r}(t) \), namely
\[
\frac{d^2 \rho}{dt^2} = M(\tilde{r}(t))\rho \tag{4.2.1}
\]
are discussed. This last equation is useful in providing both the indirect method of computing \( \partial \tilde{r}/\partial \psi \) etc. in the calculation of \( \Delta \) and stability information in the ray calculation, and more importantly it gives access to the properties of
zeros of \( \Delta \). In equation (4.2.1), when the ray and perturbation are in Cartesian components, \( \mathbf{M} \) is a symmetric, \( 3 \times 3 \) matrix whose \((i, j)\) element \( m_{ij} \) is

\[
m_{ij} = \frac{\partial U_i}{\partial x_j} = \frac{1}{2} \frac{\partial^2 N^2}{\partial x_i \partial x_j}. \tag{4.2.2}
\]

Also define the operator \( \mathbf{T}(\mathbf{r}, \dot{\mathbf{r}}) \), on the elements of a matrix \( \mathbf{P}(\mathbf{r}, \dot{\mathbf{r}}) \), where

\[
(\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t)) = \dot{\mathbf{r}}(t) = \frac{d\mathbf{r}}{dt}. \tag{4.2.3}
\]

This operator is induced by differentiation along a ray, so that if \( \mathbf{P} \) has \((i, j)\) element \( p_{ij} \), the \((i, j)\) element of \( (\mathbf{T}\mathbf{P}) \) is

\[
\frac{dp_{ij}}{dt} = (\mathbf{T}\mathbf{P})_{ij} = \sum_k \dot{x}_k \frac{\partial p_{ij}}{\partial x_k} + \sum_k U_k \frac{\partial p_{ij}}{\partial x_k}. \tag{4.2.4}
\]

Equation (4.1.2) can be integrated formally to yield, for a specified ray

\[
\mathbf{r}(t) \equiv \mathbf{r}(t; \psi_0, \theta_0) \tag{4.2.5}
\]

\[
\dot{\mathbf{r}}(t) = \mathbf{r}(0) + bt + \int_0^t (t-u) \mathbf{U}(\mathbf{r}(u)) \, du \tag{4.2.6}
\]

which can be interpreted as an integral equation for \( \mathbf{r}(t) \). Similarly, for given \( \mathbf{r}(t) \), equation (4.2.1) is formally integrable and an integral equation

\[
\dot{\mathbf{\rho}}(t) = \mathbf{\rho}(t; \psi_0, \theta_0) \tag{4.2.7}
\]

\[
\mathbf{\rho}(t) = \alpha + \beta t + \int_0^t (t-u) \mathbf{M}(\mathbf{r}(u)) \mathbf{\rho}(u) \, du
\]

for

\[
\mathbf{\rho}(t) = \mathbf{\rho}(t; \psi_0, \theta_0) \tag{4.2.8}
\]

results.

Both equations (4.2.6) and (4.2.7) could be solved by Picard iterations, respectively

\[
\mathbf{r}_k(t) = \mathbf{r}(0) + bt + \int_0^t (t-u) \mathbf{U}(\mathbf{r}_{k-1}(u)) \, du, \quad k = 1, 2, \ldots
\]

\[
\mathbf{r}_0(t) = \mathbf{r}(0) + bt \tag{4.2.9}
\]

and, with given \( \mathbf{r} \)

\[
\mathbf{\rho}_k(t) = \alpha + \beta t + \int_0^t (t-u) \mathbf{M}(\mathbf{r}(u)) \mathbf{\rho}_{k-1}(u) \, du \quad k = 1, 2, \ldots
\]

\[
\mathbf{\rho}_0(t) = \alpha + \beta t. \tag{4.2.10}
\]

Since the ray originates at \( \mathbf{S}(\psi_0, \theta_0) \) it is required that

\[
\mathbf{r}(0) = \mathbf{S}(\psi_0, \theta_0) \tag{4.2.11}
\]

\[
b = \mathbf{N}(\mathbf{r}(0)) \mathbf{n}(\psi_0, \theta_0),
\]

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and because \( \dot{\rho} \) is a perturbation either
\[
\alpha = S_\phi(\psi_0, \theta_0)
\]
\[
\beta = \left( (S_\phi \cdot \nabla N)n - (N/R_1)S_\phi \right)(\psi_0, \theta_0)
\]
for the calculation of
\[
\rho_1 = \frac{\partial \check{r}}{\partial \psi},
\]
or
\[
\alpha = S_\phi(\psi_0, \theta_0)
\]
\[
\beta = \left( (S_\phi \cdot \nabla N)n - (N/R_2)S_\phi \right)(\psi_0, \theta_0)
\]
for the calculation of
\[
\rho_{11} = \frac{\partial \check{r}}{\partial \theta}.
\]
It is noted that both the derivative initial conditions in (4.2.12) and (4.2.14) have a linear property with respect to their displacement initial conditions - in the sense that in both cases
\[
\beta = \left( (\alpha \cdot \nabla N)n - (N/R_1)\alpha \right)(\psi_0, \theta_0) \quad j = 1, 2.
\]
The derivative
\[
\frac{dr}{dt}
\]
also satisfies the perturbation equation (4.2.1).

The iterations (4.2.9) and (4.2.10) will converge uniformly in each of their components on any interval, given suitable bounds on the components of \( U \) and elements of \( M \). In this way the existence and uniqueness of the initial value problems for rays and their perturbations can be established.

But when the components of \( U \) (and so the elements of \( M \)) are analytic, it may be better to construct solutions of the integral equations (4.2.6) and (4.2.7) by their repeated partial integration. Thus the Taylor series
\[
\check{r}(t) = \check{r}(0) + bt + \left( t^2/2 \right) U(\check{r}(0)) + \left( t^3/3! \right) M(\check{r}(0))b
\]
\[
+ \left( t^4/4! \right) \left( M(\check{r}(0))U(\check{r}(0)) + (T(\check{r}(0), b)M(\check{r}(0)))b \right)
\]
\[
+ \ldots
\]
and
\[
\check{\rho}(t) = \alpha + \beta t + \left( t^2/2 \right) + \left( t^3/3! \right) \left( M(\check{r}(0))\beta + (T(\check{r}(0), b)M(\check{r}(0)))\alpha \right)
\]
\[
+ \left( t^4/4! \right) \left( (T(TM))\alpha + 2(TM)\beta + M^2\alpha \right)(\check{r}(0), b)
\]
\[
+ \ldots
\]
can be constructed.

The rays can now be assumed known as functions of $t$, and on a given ray the phase

$$\phi(t) = \int_0^t N^2(\rho(u))du$$

can be evaluated using successive partial integration. For example, in terms of the initial conditions it is

$$\phi(t) = tN^2(\bar{r}(0)) + t^2U(\bar{r}(0)) \cdot b + (2t^3/3!)(M(\bar{r}(0))b) \cdot b + |U(\bar{r}(0))|^2 + \cdots$$

(4.2.18)

4.3

The formalism from the previous section 4.2 is next used to fix the properties of the zeros of $\Delta$ (equation 4.1.10), and to connect the geometry of a constant-$\phi$ with that of a caustic surface at a singular terminal point of the former which is also a point of the latter.

At a point on a given ray, (except for the zero value) the value of $\Delta$ depends on the coordinatization of the initial surface but the location of its zero(s) is independent of that choice. Thus suppose a given ray $\bar{r}(t)$ has the singularity property at $t = t_0$ so that

$$\Delta(\bar{r}(t_0)) = 0.$$  

(4.3.1)

Let $0 < \delta < 1$ be an arbitrary but small parameter, and the initial surface $S$ be the constant-$\phi$ surface

$$\phi = \phi(\bar{r}(t_0 - \delta))$$

(4.3.2)

described in principal coordinates as

$$r = S(\psi, \theta)$$

(4.3.3)

with, for convenience

$$\bar{r}(t_0 - \delta) = S(0,0).$$

(4.3.4)

For $t_0 > t > t_0 - \delta$ the ray $\bar{r}(t)$ and the pencil of rays which contain it will be computed using equations (4.2.11) and (4.2.16) for the former, and equations (4.2.12), (4.2.14) and (4.2.17) for the others, which are viewed as perturbations of $\bar{r}$. This process will relate the required root to the geometry of $S$.

To make the calculation as direct as possible, specify the previously introduced Cartesian coordinates as $(x, y, z)$ with origin at $\bar{r}(t_0 - \delta)$, the $x$-axis in the direction of $S_\psi(0,0)$, the $y$-axis in the direction of $S_\theta(0,0)$ and the $z$-axis in the direction of $n(0,0)$. The senses of $\psi$ and $\theta$ increasing are assumed to make the coordinate system right handed. At the origin of this system, where subsequent evaluations are made, the operators

$$S_\psi \cdot \nabla = |S_\psi| \frac{\partial}{\partial x} = h_1 \frac{\partial}{\partial x}$$

etc. are the same and will be used interchangeably.
For $t_0 \geq t \geq t_0 - \delta$, and with
$$\tau = t - (t_0 - \delta)$$
the ray (4.2.16) is
$$\dot{r}(\tau + t_0 - \delta) = S(0,0) + \nabla N(0,0,1) + \tau^2 N(N_x, N_y, N_z)/2$$
$$+ N\tau^3 ((N^2)_{xx}, (N^2)_{yz}, (N^2)_{zx})/12 + \mathcal{O}(\tau^4)$$
(4.3.6)

with $N$ and its partial derivatives evaluated at $(0,0,0)$. It follows that
$$\frac{d}{dt} \dot{r}(t) = \frac{d}{dt} \dot{r}(\tau + t_0 - \delta)$$
$$= N(0,0,1) + \nabla N(N_x, N_y, N_z)$$
$$+ N\tau^3 ((N^2)_{xx}, (N^2)_{yz}, (N^2)_{zx})/4 + \mathcal{O}(\tau^3).$$
(4.3.7)

The perturbations are
$$\rho_0 = h_1(1,0,0) + \nabla h_1(0,0,1) - \frac{N h_1}{R_1}(1,0,0)$$
$$+ \tau^2 h_1((N^2)_{xx}, (N^2)_{yz}, (N^2)_{zx})/4 + \mathcal{O}(\tau^3)$$
(4.3.8)

and
$$\rho_{11} = h_2(0,1,0) + \nabla h_2(0,0,1) - \frac{N h_2}{R_2}(0,1,0)$$
$$+ \tau^2 h_2((N^2)_{yy}, (N^2)_{yz}, (N^2)_{xy})/4 + \mathcal{O}(\tau^3)$$
(4.3.9)

using (4.2.17), with evaluations at $\dot{r}(t_0 - \delta)$.

These last results then give (4.1.10)
$$(N\Delta)(\dot{r}(\tau + t_0 - \delta)$$
$$= h_1 h_2 N\left(1 - \frac{\tau N}{R_1} + \frac{\tau^2}{4} (N^2)_{xx} \left(1 - \frac{\tau N}{R_2} + \frac{\tau^2}{4} (N^2)_{yy} - \tau^2 (N^2_x + N^2_y) \right) \right)$$
$$+ \mathcal{O}(\tau^3),$$
(4.3.10)

with right hand side evaluations at $\dot{r}(t_0 - \delta))$. By assumption, the right hand side of this expression vanishes when $\tau = \delta$, and $\delta$ is small. Hence one of the factors (not $h_1 h_2 N$) in the first product must be small; assume this is the first. The root can then be expanded in powers of the necessarily small parameter
$$\frac{R_1}{N}$$
as
$$\tau = \frac{R_1}{N} \left(1 - (\frac{R_1}{N})^2 (N^2_x + N^2_y) \right) + \mathcal{O}\left((\frac{R_1}{N})^4\right)$$
$$= \delta, \text{ by assumption.}$$
(4.3.11)
Clearly at the root neither $\rho_1$ nor $\rho_{11}$ are generally zero vectors (although for this choice of initial surface, $|\rho_1|$ is $O(\delta)$), and $d\mathbf{r}/dt$ is never zero. (This distinguishes the present considerations from those of section 2.1 where conditions equivalent to $\rho_1 = 0$ or $\rho_{11} = 0$ locate the caustic surface.) The condition

$$\Delta(\mathbf{r}(t_0)) = 0$$

requires that the three vectors $\rho_1$, $\rho_{11}$ and $d\mathbf{r}/dt$ are coplanar at $t = t_0$, and their common plane (and each of the individual vectors) is tangent to the caustic surface at $\mathbf{r}(t_0)$.

Further conclusions follow. Unless the factor $(1 - \delta N/R_2)$ is also near zero, the derivative $d\Delta/d\tau$ is non-zero at $\tau = \delta$, so the zero of $\Delta$ is a simple one. (In the expected case it will be quadratic.) The limit result

$$\lim_{\delta \to 0} \frac{\delta N}{R_1} = 1$$

also follows from (4.3.8) and (4.3.11). On observing that the product $N\delta$ is, in the limit, arc length $(R - \sigma)$ measured along the ray $\mathbf{r}$ between the point $\mathbf{r}(t_0)$ on the caustic (where $R = \sigma(t_0)$) and the point $\mathbf{r}(t_0 - \delta)$ (where $\sigma = \sigma(t_0 - \delta)$), the last limit statement can be rewritten as

$$\lim_{\sigma \to R} \frac{R - \sigma}{R_1} = 1.$$  

(4.3.13)

As this limit statement (4.3.13) will be true also for rays adjacent to $\mathbf{r}(t)$ which are calculated from $S$, a patch of caustic surface can be located with respect to $S$ as either

$$\mathbf{r}_1(\psi, \theta) = S(\psi, \theta) + (R - \sigma)\mathbf{n}(\psi, \theta) + O((R - \sigma)^2)$$

$$= S(\psi, \theta) + R_1(\psi, \theta)\mathbf{n}(\psi, \theta) + O(R_1^2)$$

(4.3.14)

and then the arguments of section 2.1 can be recycled to show that, in the limit as the caustic surface is approached along a ray, the principal direction corresponding to the vanishing curvature radius on the relevant constant-$\phi$ surface is normal to the caustic surface at the ray’s point of tangency.

Moreover suppose that the plane containing the the normal $\nu$ to the caustic surface and a ray’s tangent vector at its point of tangency $\mathbf{p}$ with that surface cuts the latter in a plane curve whose radius of curvature function is $\rho$. If the principal curvature radius field $R_1$ for the constant-$\phi$ surfaces generated by the pencil of rays containing the given ray has been computed (approximately at any point, the arc length from that point to the caustic surface, by 4.3.13), then taking the limit along the specified ray

$$\lim_{\sigma \to \sigma_0} (R_1\nu \cdot \nabla R_1) = \rho(\mathbf{p})$$

(4.3.15)

- essentially the same result as (2.1.21).
4.4

The next objective is to express the geometrical optics ray solution obtained for non-constant $N$ in a form suitable for matching with the local, constant-$N$ solution that applies in the neighbourhood of a regular point of the caustic surface $r_1$, and developed in section 3.2. Thus it is supposed that on a particular ray

$$r = r(\sigma; \psi_0, \theta_0) \equiv r(\sigma); \quad r(0) = S(\psi_0, \theta_0)$$

(4.4.1)

now parametrized by arc length

$$\sigma = \int_0^t N(r(u))du$$

(4.4.2)

the geometrical optics approximation (4.1.1) is

$$H(r(\sigma)) = K(\psi_0, \theta_0) \left( \frac{\Delta(r(0))N(r(0))}{\Delta(r(\sigma))N(r(\sigma))} \right)^{1/2} \cos(\phi/\epsilon)$$

$$\equiv A \cos(\phi/\epsilon).$$

(4.4.3)

Suppose that, on the given ray, $\sigma = R$ locates a zero of $\Delta$ and hence a point of the ray’s tangency with the caustic surface, where the solution (3.2.33) is to be matched, and that the point $r(R)$ is a regular point of the caustic surface, so with two independent surface tangent vectors there. Then $\delta$ has a simple zero at $\sigma = R$, as seen in section 4.3, and

$$\lim_{\sigma \to R^-} \left( K(\psi_0, \theta_0) \left( \frac{\Delta(r(0))N(r(0))}{\Delta(r(\sigma))N(r(\sigma))} \right)^{1/2} (R - \sigma)^{1/2} \right) = l(\psi_0, \theta_0)$$

(4.4.4)

say. The problem is now to express the limit form of the amplitude

$$A \sim l(\psi_0, \theta_0)(R - \sigma)^{-1/2}$$

(4.4.5)

in a way that is suitable for the matching procedure envisaged.

Suppose the non-vanishing principal curvature radius of the constant-$\phi$ surface $\phi = \phi(r(R))$ at $r(R)$ is $R_2 > 0$; the corresponding principal direction there is tangent to the caustic surface, as has been seen. The other principal direction is normal to the caustic surface, but its principal curvature radius $R_1 = 0$. But if $R_1$ is considered to be a function of $\sigma$ on the ray $r(\sigma)$, then by the result (4.3.13) the limit statement

$$\lim_{\sigma \to R^-} \left( \frac{R - \sigma}{R_1} \right) = 1$$

(4.4.6)

applies. Next, the limit form (4.4.5) can be re-expressed as

$$l(\psi_0, \theta_0)(R - \sigma)^{-1/2}$$

$$= l(\psi_0, \theta_0) \left( \frac{1}{R} - \frac{1}{R + R_2} \right)^{-1/2} \left( \frac{1}{R} - \frac{1}{R + R_2} \right)^{1/2}$$

$$= k(\psi_0, \theta_0) \left( \frac{1}{R} - \frac{1}{R + R_2} \right)^{-1/2},$$

(4.4.7)
where the definition is made

\[ k(\psi_0, \theta_0) = l(\psi_0, \theta_0) (\frac{1}{R} - \frac{1}{R + R_2})^{1/2}, \quad (4.4.8) \]

bearing in mind that R and R_2 also depend on (\psi_0, \theta_0). In this way the expression (4.4.7) can be identified with the limit form of the amplitude in the constant-N calculation (3.1.14) (\sigma < R_1 < R_2) namely

\[ \left( K(R_1 R_2)^{1/2}((R_1 - \sigma)(R_2 - \sigma))^{-1/2}\right)(\psi_0, \theta_0) \]

\[ \sim \left( K((R_1 - \sigma)(\frac{1}{R_1} - \frac{1}{R_2}))^{-1/2}\right)(\psi_0, \theta_0) \quad (4.4.9) \]

as \( \sigma \to R_{1-} \) if

k is identified with K
R is identified with R_1
(R+R_2) is identified with R_2.

Thus all but two of the essential parameters for using a constant-N, near caustic surface solution (3.2.33) to describe the same situation the non-constant-N case are identified. To complete the process, it is noted that the combination

\[ R_1 R_{1e}/h_1 = \mu \]

in the constant-N solution must be replaced with

\[ \lim_{\sigma \to R} \left( (R - \sigma)\nu \cdot \nabla (R - \sigma) \right) = \lim_{\sigma \to R} \left( (R - \sigma)\nu \cdot \nabla R \right) \]

\[ = \lim_{\sigma \to R} \left( R_1 \nu \cdot \nabla R_1 \right), \quad (4.4.10) \]

where the limit is taken along the ray r(\sigma), and the normal \( \nu \) to the caustic surface is at this ray's point of tangency with it. Finally, the small parameter \( \epsilon \) in the expression (3.2.33) must be replaced with \( \epsilon/N(R) \).

References