ABSTRACT

A class of direction field singularities is defined for ordinary differential equations. The usefulness of the classification is demonstrated through examples.
A rather natural, simple evolution from the classical Poincaré-Bendixon theory of first order, ordinary differential equations is described below. The technique is useful for computing local approximations to solution trajectories for second and higher order ordinary differential equations in certain circumstances, but its description is incomplete since, for want of general theorems, applications must be subject to a posteriori verification.

The matching of such local('inner') approximations to 'outer' approximations using standard techniques (see e.g. Cole [1]) may be helpful in those cases where the latter are not uniformly applicable, thereby providing an alternative, equivalent method to the P.L.K. (Tsien [2]) routine for rendering solutions uniformly valid.
Consider the ordinary differential equation governing \( f(x) \)

\[
\frac{f^{(n)}}{B(f^{(n-1)}, \ldots, f(0), x)} = H
\]

where \( f^{(j)} \) is the \( j \)th derivative

\[
f^{(j)} = \frac{d^j f}{dx^j}
\]

\[
f(0) = f
\]

of the real valued function \( f \) of the real variable \( x \). Further, suppose that there is a 'point' \( P (f^{(n-1)}_0, \ldots, f(0)_0, x_0) \) with

\[
|x_0| + \sum_{j=0}^{n-1} |f^{(j)}_0| < \infty
\]

at which both

\[
A(f^{(n-1)}_0, \ldots, f(0)_0, x_0) = 0
\]

and

\[
B(f^{(n-1)}_0, \ldots, f(0)_0, x_0) = 0
\]
but at which not all $A_j, B_j$ are zero ($j = 0, \ldots, n-1$) where

$$A_{j+1} = \frac{\partial A}{\partial f(j)} \bigg\vert_p$$

$$A_0 = \frac{\partial A}{\partial x} \bigg\vert_p$$

etc.

If at $P$ the functions $A$ and $B$ are regular in all their arguments, then a local approximation to the right hand side $H$ of the differential equation (1) is $H_1$ where

$$H_1 = \frac{\left[ \sum_{0}^{n-1} (f(j) - f_0(j))A_{j+1} \right] + (x - x_0)A_0}{n-1}$$

$$\left[ \sum_{0}^{n-1} (f(j) - f_0(j))B_{j+1} \right] + (x - x_0)B_0$$

Suppose $f(x)$ satisfies the differential equation (1) and passes through the point $P$, then $H$ can be further approximated as

$$H = \frac{(x - x_0)[(\sum_{0}^{n-2} f_0(j+1)A_{j+1} + A_0] + (f(n-1) - f_0(n-1))A_n}{(x - x_0)[(\sum_{0}^{n-2} f_0(j+1)B_{j+1} + B_0] + (f(n-1) - f_0(n-1))B_n}$$

$$\frac{\alpha \varphi + A_n \varphi}{\beta \varphi + B_n \varphi}$$

(7)
where the definitions

\[ a = \left( \sum_{j+1}^{n-2} f_0^{(j+1)} A_{j+1} \right) + A_0 \]

\[ \beta = \left( \sum_{j+1}^{n-2} f_0^{(j+1)} B_{j+1} \right) + B_0 \]

\[ \mathcal{J} = f_0^{(n-1)} - f_0^{(n-1)} \]

\[ \xi = x - x_0 \]

have been introduced. It is assumed that the fraction (7) is irreducible, that is

\[ \alpha B_n - \beta A_n \neq 0 \]  \hspace{1cm} (9)

Definition

The ordinary differential equation (1) is said to have a simple singular point at \( P \) \( (f_0^{(n-1)}, \ldots, f_0^{(0)}, x_0) \) if its solution trajectories in a (possibly deleted) neighbourhood of that point are in one to one correspondence with, and can be approximated locally by, the solution trajectories of the equation
\[
\frac{dF}{d\xi} = \frac{\alpha_x + A_n F}{\beta_x + B_n F} \tag{10}
\]

with the correspondence

\[
F(\xi) \approx f^{(n-1)}(x) - f^{(n-1)}_0 \tag{11}
\]

The aim is now to show by examples that the idea has some validity and usefulness. The examples, which are drawn from physical and artificial problems, have been selected as the results obtained through the present method can be verified by other means.

Example 1

The propagation of a cylindrical shock wave in a compressible gas can be described from solutions of the boundary value problem

\[
f^{(2)} = -\frac{f^{(1)}}{x} \left\{ \frac{1 - (\gamma - 1)[f^{(0)} - xf^{(1)} + \frac{1}{2}(f^{(1)})^2]}{1 - (\gamma - 1)[f^{(0)} - xf^{(1)} + \frac{1}{2}(f^{(1)})^2]} - (x - f^{(1)})^2 \right\} \tag{12}
\]

\[
f^{(1)}(\varepsilon) = \varepsilon \quad a
\]

\[
f(M) = 0 \quad b
\]

\[
f^{(1)}(M) = \frac{2}{\gamma + 1} (M - \frac{1}{M}) \quad c
\]

Particular interest is in the case $0 < M - 1 \ll 1$, $0 < \varepsilon \ll 1$ and the exercise is to find an approximate $M(\varepsilon)$ relation for the (over prescribed) problem 12, 13.
The set of simple singular points (S.S.P.) of interest are those in the plane (in x, f(0), f'(1) space) f' = 0 which form the line

\[ 1 - (\gamma - 1)f(0) - x^2 = 0. \]

The conditions further suggest that the S.S.P. for study will be near \( x = 1, f(0) = 0 \); this location will be assumed in approximating the coefficients \( a, \beta, A_n B_n \).

If the obvious identifications are made, the locally approximating equation is

\[ \frac{dF}{d\xi} = -\frac{(1 - (\gamma - 1)f(0))F}{(\gamma + 1)F - 2x_0^2 \xi} \quad (14) \]

but if its coefficient are approximated as suggested above it becomes

\[ \frac{dF}{d\xi} = -\frac{F}{(\gamma + 1)F - 2\xi} \quad (15) \]

The latter equation (15) has a one parameter family of solutions which are conveniently expressed as

\[ \frac{F - \xi/(\gamma + 1)}{F^2} = \beta \quad (\text{const}). \quad (16) \]
and the two branches of each member of the family are given by the solution of a quadratic as

\[ F = \frac{1 \pm \sqrt{1 - \frac{4\beta \varphi}{\gamma+1}}}{2\beta} \]  

(17)

The boundary condition \( c \) can be satisfied approximately if, (bearing in mind \( 0 < (M-1) < 1 \))

\[ \frac{4}{\gamma+1}(M-1) = 1 + \sqrt{1 - \frac{4\beta (M-1)}{\gamma+1}} \]

that is, if

\[ \beta = \frac{3}{16} (\gamma+1)(M-1)^{-1} \]  

(18)

To develop the solution to equation (12) away from the point \( x = M \) (the outer solution), the assumption is made that the linear approximation

\[ (1-x^2)f''(2) = -\frac{f'(1)}{x} \]  

(19)

to equation (12) is applicable. The assumption can be verified a posteriori. The approximating equation (19) has solution
\[ f(1) = K \frac{\sqrt{1-x^2}}{x} \] \hspace{1cm} (20)

and it matches the local approximation (17) if

\[ K = (2(\gamma + 1)\beta)^{-\frac{1}{2}} \] \hspace{1cm} (21)

But the boundary condition (13 a) remains to be satisfied and it requires that

\[ K = \varepsilon^2 \] \hspace{1cm} (22)

then equations (17), (21), (22) give the \( M(\varepsilon) \) relationship to be approximately

\[ M = 1 + \frac{3}{8} (\gamma + 1)^2 \varepsilon^4 \] \hspace{1cm} (23)

Comment

The result (23) was originally obtained by Lighthill [3] by a rather more elaborate computation, which depended on the availability of an closed form solution to an equation which uniformly approximated equation (12). Lighthill [3] was also able to give a proof of the validity of the result (23). The present technique will obviously work for analogous problems in spherical and conical flows.
Example 2

A simple example arises from a study of the confluent hypergeometric equation

\[ f^{(2)} = - \frac{((\gamma - x)f^{(1)} - \alpha f^{(0)})}{x} \]  

(24)

The latter has a line of S.S.P. (in \( x, f^{(0)}, f^{(1)} \) space) in the plane \( x = 0 \) and with

\[ f^{(1)}_0 = \frac{a}{\gamma} f^{(0)}_0 \]  

(25)

The locally approximating equation is

\[ \frac{dF}{d\xi} = - \frac{[\gamma F - f^{(0)}_0 \xi \frac{a}{\gamma} (1 + \alpha)]}{\xi} \]  

(26)

and so the approximations suggested are

\[ f^{(1)}(x) = c_1 x^{1-\gamma} f^{(0)} + x \frac{a(1 + \alpha)}{\gamma(\gamma + 1)} f^{(0)}_0 \]

\[ f^{(0)}(x) = c_1 x^{1-\gamma} f^{(0)} + x \frac{a}{\gamma} f^{(0)}_0 + x^2 \frac{a(1 + \alpha)}{\gamma(\gamma + 1)} f^{(0)}_0 \]  

(27)

The latter being recognizable as an approximation to a linear combination of two independent solutions of the confluent hypergeometric equation.
Comment

Although, for internal consistency of the method it is required that $-\gamma > 0$, it is arguable that the method gives a satisfactory result even for positive values of $\gamma$ in an interval which excludes $x = 0$.

Example 3a

The non-linear equation

$$f(1) = -\frac{f(0)(1 + (f(0))^2)}{ax}$$

has the exact solution

$$f(0) = \pm \left( \frac{1}{c^2|\frac{1}{\alpha} - 1} \right)^{1/2}$$

while the result obtained by examining the putative S.S.P. at $f(0) = 0, x = 0$ is

$$F = \frac{K}{|x|^{1/\alpha}} \approx f(0).$$
Comment

For \( a \) negative, the above expression is a valid approximation to the family of solution trajectories to the original equation. For positive values of \( a \) the method is not internally consistent, but it does provide an approximation to the leading term of the exact solution(s) when it is expanded for \( c|x|^{2/a} \gg 1 \). Thus \( f^{(0)} = 0 \), \( x = 0 \) still falls within the ambit of the definition of the S.S.P.

Example 3b

The boundary layer on a paraboloid of revolution in a viscous fluid flow, well downstream from its nose, can be described using the solution of the ordinary differential equation

\[
 f^{(3)} = -\frac{f^{(2)}(1 + f^{(0)})}{x} \quad (28)
\]

which satisfies the boundary conditions

\[
 \begin{align*}
 f^{(0)}(R) &= 0 & \text{a} \\
 f^{(1)}(R) &= 0 & \text{b} \\
 f^{(0)}(x) &\sim x - \beta & x \to \infty & \text{c}
\end{align*}
\]

Here \( R \) is the Reynolds number of the flow, and the interesting case for the present is \( 0 < R < 1 \). The relation between \( \beta \) and \( R \) is sought.
The differential equation (28) has an apparent S.S.P. at $x = 0$, $f'(2) = 0$, and so the local approximation is

$$\frac{dF}{d\xi} = -(1 + f'(0)) \frac{F}{\xi}$$

whose solution is

$$F = K \xi^{-\left(1 + f'(0)\right)}$$

where $f'(0)$ and $K$ are constants which remain to be determined.

Thus the suggested approximation to $f'(2)$ is

$$f'(2) = K x^{-\left(1 + f'(0)\right)}.$$  \hfill (32)

Of course, for internal consistency it would be required that the inequality

$$-(1 + f'(0)) > 0$$

hold, but for the present simply assume that the approximation (32) is useful in some interval $1 \gg x \gg R$.

The outer approximation is obtained by dominating an iteration of the governing equation (28) using the outer boundary condition (29c). Thus the result is found

$$h(2) = C x^{-1 + \beta} e^{-x}$$

(33)
approximately, and this matches the previous (32) if

\[ \beta = -f_0(0) \]

\[ c = K. \]  \hspace{1cm} (34) \]

Here it is appropriate to make two observations. The first one is that it is not permissible to twice integrate the uniform approximation to \( f(2) \) namely

\[ f(2) = c x^{-1+\beta} e^{-x} \]

from (say) the terminal \( R \), and obtain a valid approximation to \( f(0) \) at the terminal \( \infty \) (or vice versa). The second observation is that the expression

\[ 1 = R f'(2) \left( \frac{U}{t} \right) \left( f(0)(t) \right) dt \]

\[ \int_{R}^{\infty} e^{-t} \left( \frac{U}{t} \right) \frac{du}{u} \]

suggests the inequality

\[ f'(2)(R) \gg 1, \]

since \( R \) is assumed to be small.
If the outer approximation to $f^{(2)}$ is twice integrated from the outer terminal, the approximation follows

$$f'(0) = x - \beta + c \left\{ e^{-x} x^\beta - (x - \beta) \Gamma(\beta, x) \right\}, \quad (35)$$

where the incomplete gamma function is defined as

$$\Gamma(\beta, x) = \int_x^\infty e^{-t} t^{-1+\beta} \, dt.$$ 

The corresponding inner approximation is obtained by twice integrating the approximate $f^{(2)}$ from the inner terminal. If it is assumed (and verified a posteriori) that $\beta$ is small, this approximation is

$$f'(0) = c \{(x \ln x - x) - x \ln R + R\}. \quad (36)$$

Upon noting the approximation to the incomplete gamma function

$$\Gamma(\beta, x) = -0.5772 \ldots -\ln x \ldots$$

which is valid for $1 \gg x \gg \beta$, the matching of the two approximations $(35),(36)$ is accomplished if

$$\beta = c = -\frac{1}{|\ln R|} + O(\ln R)^{-2} \quad (37)$$
Comment

The result (37) was first obtained by Miller [4] by essentially similar analysis to the above, but without explicit recourse to the idea of an ordinary singular point.

Even though the method is not internally consistent - $F(0)$ is unbounded - it provides an adequate result in a deleted neighbourhood of the origin.
REFERENCES


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