Assignment 2: Solutions to the TWO questions as described below are due to be handed in at the class 11am Tue 18th Aug.

- If the 4th digit of your student number is \( n \), hand in a very carefully written complete solution, by hand calculation, of your answer to question numbered \((1 + (n \mod 8))\).
- Answer the last question (in group work with between 1 and 4 of your friends).

You may find useful for parts of this assignment;

- the Matlab Lab Sheets DE16h on the heat equation (and DE16w on the wave equation), and the CAA assessed quizzes on PDEs;
- Chpt 25 from the HELM (‘Helping Engineers Learn Mathematics’) collection of pdfs accessible via the ENVE3605 web pages.

1. Find the general solutions of
   \[
   \begin{align*}
   (i) & \quad \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} = 3 \exp(x + \alpha t) \quad (\alpha \neq -\frac{1}{2}); \\
   (ii) & \quad x \frac{\partial u}{\partial x} + 3t \frac{\partial u}{\partial t} = tx^{-\alpha} \quad (\alpha \neq 3); \\
   (iii) & \quad 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial t} + \alpha u = 0; \\
   (iv) & \quad 3x \frac{\partial u}{\partial x} + 2t \frac{\partial u}{\partial t} + \alpha u = 0
   \end{align*}
   \]
   for all values of the constant \( \alpha \) except the values indicated.

2. Verify that the general solution of
   \[
   \tan(x) \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0
   \]
   is \( u = f(\xi) \) where \( f \) is an arbitrary function of \( \xi = (\sin(x))/y \).
   Determine the solution which takes the value \( u = 1/(\cos(x)^2) \) on \( y = \cos(x) \).

3. Verify that the general solution of
   \[
   (1 - x^2) \frac{\partial u}{\partial x} + 2y \frac{\partial u}{\partial y} = 0
   \]
   is \( u = f(\eta) \) where \( f \) is an arbitrary function of \( \eta = y(1-x)/(1+x) \).
   Determine the solution \( u(x, y) \) which takes the value \( u = 1/(1+x) \) on \( y = 1 \).

4. Consult wikipedia
   
   
   for the definition, and properties, of the complementary error function \( \text{erfc} \).
   In particular, note the formula for its derivative and a consequence that the ratio of the second and first derivatives of \( \text{erfc}(z) \) is proportional to \( z \).
   A semi-infinite solid \( x > 0 \) is initially at temperature zero. The face \( x = 0 \) is kept at a temperature \( U_0 \) for all \( t > 0 \). The governing equation is
   \[
   \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}
   \]
Verify that

\[ u(x, t) = U_0 \text{erfc}(x/(2\sqrt{\kappa t})) \]

solves the pde, the initial conditions and the boundary conditions.

Use Matlab \texttt{ezcontour} or \texttt{contour} to draw, in the positive quadrant of \((x, t)\)-space, contours of the temperature \(u\). Now describe the contours in words, and explain why one really doesn’t need Matlab this time for understanding the general shape of the contours.

Optional. Use Laplace transforms in \(t\) to derive the solution given above.

5. Consider the solution \(u(x, y)\) of Laplace’s equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

in the half-space \(-\infty < x < \infty, y > 0\), which satisfies the boundary conditions \(u(x, 0) = f(x)\) and \(u \to 0\) as \(x^2 + y^2 \to \infty\).

(By Fourier transforming in \(x\) and using the convolution theorem) it can be shown that

\[
u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\hat{x})}{(x - \hat{x})^2 + y^2} d\hat{x}
\]

(i) Given that

\[
f(x) = \begin{cases} u_0 & \text{on } |x| \leq a, \\ 0 & \text{elsewhere.} \end{cases}
\]

where \(u_0\) and \(a\) are positive constants, deduce that

\[
u(x, y) = \frac{u_0}{\pi} \left( \tan^{-1} \left( \frac{x + a}{y} \right) - \tan^{-1} \left( \frac{x - a}{y} \right) \right)
\]

(ii) Derive the alternative expression

\[
u(x, y) = \frac{u_0}{\pi} \tan^{-1} \left( \frac{2ay}{x^2 + y^2 - a^2} \right)
\]

for the solution.

(iii) Use Matlab \texttt{ezcontour} or \texttt{contour} to draw in the positive half-plane, \(y > 0\), contours of the temperature \(u\). Now describe the contours in words, and explain why one really doesn’t need Matlab this time for understanding the general shape of the contours.

(iv) Show that \(u(x, y) = u_0/2\) on the semicircle \(x^2 + y^2 = a^2, y > 0\).

6. (i) Use a Fourier sine series to find the solution \(u = u(x, y)\) of Laplace’s equation in the square \((0, \pi) \times (0, \pi)\), that satisfies the boundary conditions

\[
u(0, y) = u(\pi, y) = u(x, \pi) = 0 \quad \text{and} \quad u(x, 0) = x.
\]

You may use results on p18 of the Fourier Methods handout.

Plot the solution surface \(z = u(x, y)\) using the \texttt{surf} command in Matlab.

(ii) Using (i) find the solution \(u_m = u_m(x, y)\) of Laplace’s equation in the same square as in part (i), which satisfies the boundary conditions

\[
u(0, y) = y \quad \text{and} \quad u(\pi, y) = u(x, 0) = 0 \quad \text{and} \quad u(x, \pi) = x.
\]

(Hint. Use superposition.)
7. Suppose that \( u = u(x, t) \) satisfies the diffusion equation
\[
\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}
\]
on the interval \( 0 \leq x \leq \ell \). The boundary conditions are
\[
u(0, t) = u(\ell, t) = 0 \text{ for all } t \geq 0.
\]
You are given that the most general solution \( u(x, t) \) obtained by separation of variables
is the Fourier series
\[
u(x, t) = \sum_{n=1}^{\infty} C_n \exp \left( - \frac{n^2\pi^2 \kappa t}{\ell^2} \right) \sin \left( \frac{n\pi x}{\ell} \right)
\]
where \( C_n \) are arbitrary constants.
(i) Determine \( u(x, t) \) explicitly when the initial condition \( u(x, 0) = x^2 \) is imposed.
(ii) Plot the solution surface \( z = u(x, t) \) using the \texttt{surf} command in Matlab.

8. Read the document ‘A system of 1st order PDEs modelling sunlight’s effect on algal growth’. Use Matlab \texttt{ezplot} or \texttt{plot} commands to re-create the plot given on the last page of that document.

9. (For group work. Max. size of any group is 5.) Write a short account (about one page) of any pde problem you have encountered in any Environmental Engineering application.
   - Clear references to easily available sources (and I allow wikipedia) are essential.
   - Talk with others in the class outside your group and make sure your application is different from theirs.
   - You are not asked to solve the pde. However you should classify what sort of pde problem it is – elliptic, parabolic or hyperbolic (or other?). You should also state what sort of boundary and/or initial conditions would make the pde problem well-set.