Intro Lec 01

Topics linear ODE
Topics nonlinear ODE
Topics Fourier Series and linear ODE

(o) d.e.s

Today’s lecture is notes, Chapt 1
Assignments and CAA
\( y' = Ay \)

For now, this screen is an aside
First order d.e. examples
\( \dot{P} = kP \): Perth’s population
\( \dot{P} = kP \): Perth’s population, ctd

A nonlinear example
\( \dot{P} = kP^r \) with \( k > 0 \) and \( r > 1 \)

World population
World population, ctd
Population model d.e.s

Logistic Eq
\( \dot{P} = kP^r \) with \( k < 0 \) and \( r < 0 \)
Topics linear ODE

- d.e.s - revision but steering towards 2nd year work on
- matrix eigenvalues and matrix exponentials
- 2 d examples
  - `dsolve` (and use of `ode45` for numerics), phase portraits, and
  - how linear theory helps with understanding nonlinear d.e.
Topics nonlinear ODE

- equilibria
- linearizing to study the small disturbances close to equilibria leads to matrix exponentials again
- stability of equilibria for some nonlinear systems the competing species example.
Topics Fourier Series and ODE

- orthogonality in function spaces
- Fourier Series,
- periodic responses to periodically forced d.e.s

Actually, you should have seen all these topics in 1st or 2nd year (and others, e.g. Laplace transforms, as well).
I have assembled the handout notes to contain a few Environmental Engineering examples. I’m prepared to add other examples if you tell me what you think would fit nicely into the story.
There are printed notes.

A comment on the notes.
Some is revision (of material treated in earlier years). You are expected to study them if there are gaps in your knowledge. The lectures are meant to highlight important items, and to make links between the different parts.

Today: Chapt 1.
Unsurprisingly, the next lecture will be Chapt 2 (on 2nd order d.e.s).
Assignments and CAA

Assignment 1 has you revising these d.e.s and cross-checking your answers against just asking Matlab dsolve to solve them. The Computer Aided Assessment (CAA) covers similar material with a similar due date, and doing it should help with the lectures and with the assignments. The marks for the CAA are pretty small, and the penalties for wrong answers so little (often about 10% loss of available marks per wrong attempt) that learning through multiple attempts is OK.
Interested in linear d.e.s and how they can help us learn about nonlinear ones, and about the real world.
All d.e.s are easy enough to approximate numerically.
Matlab ode23, ode45, etc..
However, I won’t do numerics - and the methods underpinning them - as these belong in the numerical methods part of the unit.
Easiest example (linear):
\[ y' = Ay \] is solved by \[ y(t) = \exp(At)y(0) \]
\( A > 0 \) solution gets bigger and bigger in magnitude as \( t \) increases.
\( A < 0 \) solution gets smaller and smaller in magnitude as \( t \) increases.
For now, this screen is an aside

In first year you learnt that the scalar equation
\[ y' = Ay \] is solved by \[ y(t) = \exp(At)y(0). \]

In second year you learnt that the system of equations
\[ y' = Ay \] is solved by \[ y(t) = \text{expm}(At)y(0). \]
Here \( A \) is a square matrix.
(\text{expm} is a Matlab command too.)
All this is eminently memorable. We will revise it in a later lecture.
First order d.e. examples

Let’s start with examples of a single first order d.e.
In MATH1020 you treated

- direction fields for these,
- exact solution for
  - separable d.e.s
  - linear d.e.s
\[ \dot{P} = kP \]  
Perth’s population

Solved: 
\[ P(t) = \exp(kt)P(0) \]

See Fig 1.

Figure: Population data for Perth, 1910-2010
\[ \dot{P} = kP: \text{Perth’s population, ctd} \]

\[ \log(P(t)) = (kt) + \log(P(0)) \]

See Fig 2.

**Figure:** A line fitted to the log of Perth’s population data, 1910-2010
A nonlinear example

Positive solutions of $\dot{P} = kP^r$.  
$r = 1$ just treated.  
Treat $r > 1$ and $r < 1$ separately.
\[ \dot{P} = kP^r \text{ with } k > 0 \text{ and } r > 1 \]

\[
P(t) = ((r - 1) \left( \frac{P(0)^{1-r}}{r - 1} - kt \right))^{(1/(1-r))}
\]

\[
P(t) = \frac{1}{\frac{1}{P(0)} - kt} \quad \text{when } r = 2
\]

These solutions ‘blow up in finite time’ (as opposed to the \( r = 1 \) exponential growth which takes infinitely long).

E.g. in the \( r = 2 \) case the solution goes infinite when \( t \) increases up to \( 1/(kP(0)) \).
World population

Back in the early 1990s, there was an alarmingly good fit to ‘super-exponential’ growth. If one looks at more recent data it appears that we are near the inflection point of some sort of modified logistic growth model.
See Fig 3
World population, ctd

Figure: World population data, 1000-1998
Population model d.e.s

In Environmental Eng., the populations may well be biological ones, e.g. in water resources, water quality, d.e.s involving algae and so on. The d.e.s can get to be moderately elaborate. Returning to simple d.e.s, you would also have met the logistic d.e.

\[ \dot{P} = kP(P_{\text{inf}} - P) \]

in 1st year as yet another example of a separable 1st order d.e.. There is a simple formula for its solution. Use Matlab’s dsolve if you are struggling to do the integrals to get to its solution.
**Logistic**

\[ y' = Ay \]

For now, this screen is an aside

**First order d.e. examples**

\[ \dot{P} = kP \]

Perth's population

\[ \dot{P} = kP \]

Perth's population, ctd

**A nonlinear example**

\[ \dot{P} = kr \]

with \( k > 0 \) and \( r > 1 \)

**World population**

**World population, ctd**

**Population model d.e.s**

**Logistic Eq**

\[ \dot{P} = kr \]

with \( k < 0 \) and \( r < 0 \)

**StoppingTime Pic**

**d.e.s of various orders**

**Recognizing linear d.e.s**

The handout notes, Revision d.e.s 1

**Direction Fields**

**Direction Fields, ctd**

**Example:**

\[ y' + y / 2 = 3 / 2 \]

**exact1**

**code for exact1, 1**

**exact2**

Figure:

Solutions of the logistic equation, \( P_{\inf} = 1 \) and \( k = 1 \)

Grant Keady

**ENVE3605 Intro, ODEs**
Again consider positive solutions. These now decrease with time (as \( k < 0 \)). DEs of this kind also occur - though having nothing to do with populations as far as I’m aware. I encountered the d.e. in a mechanics sum: Euler’s disk, and its stopping in finite time. The simple d.e. describing the angle \( \alpha \) to the horizontal is

\[
\dot{\alpha} = -\frac{\epsilon}{\alpha^n}
\]

with \( \epsilon \) a positive quantity. The solution is

\[
\alpha(t_0)^{n+1} - \alpha(t)^{n+1} = (n + 1)\epsilon(t - t_0)
\]

The solution reaches zero at

\[
t_1 = t_0 + \frac{\alpha(t_0)^{n+1}}{(n + 1)\epsilon}.
\]

In the case \( n = 2 \), the finite-time stopping is shown in Figure 5.
Figure: Plot of $\alpha(t) = (1 - t)^{1/3}$. $\varepsilon = 1/3$, $t_0 = 0$, $\alpha(0) = 1$. 
In various engineering units and in maths from 1st year on you have seen mass-spring oscillator sums. 2nd order. linear
It is too nice and too important a problem for me to ignore. I hope I will be saying things you have already met. It is good to revisit maths you have already seen with Matlab around to draw pictures. Hopefully you will be re-assured by it. Some more introductory items before the ‘linear oscillator’...
### Recognizing linear d.e.s

Notation here: $y'$ means $\frac{dy}{dt}$

<table>
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<th>if linear homogeneous?</th>
<th>if linear const.coefft?</th>
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<td>Y</td>
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<tr>
<td>$y' - y(1 - y) = 0$</td>
<td>1</td>
<td>N</td>
<td></td>
<td></td>
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<tr>
<td>$my'' + cy' + ky = 0$</td>
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<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$my'' + cy' + ky = f(t)$</td>
<td>2</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
</tr>
</tbody>
</table>
The handout notes, Revision d.e.s 1

Chpt 1, 1st order
Treated in 1st year
  ▶ linear (integrating factor method=variation of parameters)
  ▶ separable
  ▶ in general, can’t find formulae for soln
    ode23, etc. numeric soln always available.
    Basic idea for numerics visible in ‘direction fields’

See the handout notes, near the picture we show next.
Direction Fields

Figure: The arrow is in the direction \([c, s]\) where \(f'(x) = s/c\). \(y' = f\) is solved by \(y = f(x)\).
Direction Fields, ctd

There are many programs for producing Direction Fields. They exist on graphing calculators (e.g. TI85). When I lectured 1st year I had a Java Applet from somewhere that did the job. With engineers, and knowing that engineers use Matlab, there are a couple of approaches.

- There is a Matlab GUI called dfIELD7 where one enters the d.e. and clicks on the starting points. In a later lecture I will demo a similar program pplane7. (Can find with google.)

- It is pretty easy to generate them though some of the code meshgrid to produce the mesh over which to put the arrows and quiver to draw lots of arrows belongs after, or in, your real introduction to Matlab. If you chose to use them sooner, just adapt the code I give, though you can survive for now without them.
  (You need dsolve though to do Assignment 1.)
Example: \( y' + y/2 = 3/2 \)

Ridiculously easy.
First solve the homogeneous (unforced d.e.)
\( y' + y/2 = 0 \) is solved by \( y_h = \exp(-t/2) \).

Next find a particular solution. The forcing is constant, the d.e. is constant coefficient. The ‘method of undetermined parameters’ says look for a particular solution that is a bit like the forcing, here a particular solution which is a constant.

\[ y_p = 3 \]

Finally, the general solution is \( y_p + Cy_h \) for any number \( C \).

Let’s use this example to see a direction field and to see Matlab doing a bit.
Figure: Direction field for $y' + y/2 = 3/2$. Really should have labelled the axes. Do in Matlab to show it better.
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StoppingTime Pic

d.e.s of various orders

Recognizing linear d.e.s

The handout notes, Revision d.e.s

Direction Fields

Direction Fields, ctd

Example:

\[ y' + \frac{y}{2} - \frac{3}{2} = 0 \]

exact1

code for exact1, 1

```matlab
% Matlab Symbolic Toolbox   FILE exact1mtl
syms y x y0
ode='Dy +y/2 -3/2';
% Set the initial condition to y(0)=y0 and solve:
initsol=dsolve(ode,'y(0)=y0','x')
y1=inline(vectorize(initsol),'x','y0')
% give Matlab some starting points. E.g.
% init y0 from -1 to 1 in steps of 1
% The following will draw the integral curves
figure; hold on
x = 0: 0.2 : 5;
for y0val =-1:1
    plot(x,y1(x,y0val))
end
axis tight; ylabel('y')
```

Grant Keady

ENVE3605 Intro, ODEs
code for exact1, 2

% You can plot, on same plot, 
% direction fields and integral curves.
[X,Y] = meshgrid(0:0.5:5, -1:0.25:3);
S = 3/2 -Y/2;
L = sqrt(1 + S.^2);
quiver(X,Y, 1./L, S./L,0.5)
axis tight
% Label axes and write a title
xlabel('x'); ylabel('y');
title(' Some solutions of Dy +y/2 -3/2=0')
hold off
What do we learn from this?

Well really the picture with all the little arrows, gives all the info on the d.e..

The solutions all want to go to the constant function $y_p = 3$ as time gets big.

Doing this with the ridiculously easy d.e. is meant to show you that the exact analytical solutions and the direction field do line up nicely.
Today’s lecture is notes, Chapt 1
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d.e.s of various orders
Recognizing linear d.e.s

The handout notes, Revision d.e.s 1
Direction Fields
Direction Fields, ctd
Example:
\[ y' + \frac{y}{2} = \frac{3}{2} \]

exact2

Figure: See printed notes. \[ yy' + (1 + y^2) \sin(x) = 0. \]
What next, for you?

Study the notes.
Do the Assignment and the CAA. They treat the material I’m lecturing on and other items in the notes.

This is a class with a small enrolment.
I’m ready to help you with your questions, if you come in groups of 2 or 3 to see me in my room 2.32.
It is only be you doing the work that you will learn.
All this d.e. work makes sense and is useful.

Change of venue for the lecture classes this week.
They are moved to MATH:MCL
(Maths Computer Lab, south-east corner.)
Agenda: Getting started with Matlab dsolve and questions on Assignment 1.