ENVE3605
2nd order ODEs

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Review of 1st year linear algebra

Vector Spaces

‘vector space’

\( \mathbb{R}^n \) example

Continuous functions, function space example
The handout notes, Revision d.e.s 2

Read the handout. Much of it should be revision. Maths can be taught in a strictly logical one-thing-after-another order. The down-side to this is that the motivation for many topics isn’t there when they are presented. The handout notes are closer to the logical order than in the lecture presentation which will empahise motivating items, etc.. Listen in lectures as I push the line ‘vector spaces are good for understanding d.e.s’. (And it is easiest to show this with linear d.e.s first.) There are ‘big picture’ aspects – vector spaces, linearity. There are also little details, what solutions to particular problems are really like, formulae, plots, etc., etc.. They both matter.

As a lecturer, my guess is that my main job in lectures is to try to get the ‘big picture’ ideas over. The assignments and Matlab work tend to emphasise the details of ‘what are the solutions’ ,... ‘what do the solutions look like’.
homogeneous/nonhomogeneous

Reminder/revision
The linear d.e.
\[ y'' + P_1y' + P_0y = 0 \]  \((H)\)
is said to be homogeneous or unforced.

The linear d.e.
\[ y'' + P_1y' + P_0y = f \]  \((N)\)
is said to be nonhomogeneous or forced.

(Our example later will be with a constant coefficient d.e., i.e. \(p\) and \(q\) constant in time, and with a sinusoidal forcing, e.g. \(f(t) = \cos(\omega t)\).)
An easy numerical example, (H)

(This is given a bit later in printed notes.)

\[ y'' + 2y' + 5y = 0, \quad (H) \]

a homogeneous d.e., has real solutions

\[ y_h(t) = C_1 \exp(-t) \cos(2t) + C_2 \exp(-t) \sin(2t) \]

for any real constants \( C_1, C_2 \).

In other words, the solutions of \((H)\) form a (function) vector space whose basis is \{\( \exp(-t) \cos(2t), \exp(-t) \sin(2t) \}\).

We will soon say little more about the term ‘vector space’.
Matlab solving (H)

Routine calculation is automated, e.g. in Matlab:

```matlab
syms y t
yh = dsolve('D2y+2*Dy+5*y=0','t')
% yh =
% C1*exp(-t)*cos(2*t)+C2*exp(-t)*sin(2*t)
```

Do the plots of the solution in Matlab (if time).
Solve i.v. problems.
Figure: A solution of $y'' + cy' + y = 0$. (Has $y(0) = 0$, $y'(0) = 1$.) $c = 0$ would give a sinusoidal solution. $c > 0$ small (underdamped) shown.
Various plots

Different situations.

- lightly damped (see oscillations, previous example)
- critically damped (the damping is such that if it were to be any less you would have oscillations)
- overdamped: no oscillations. Roots of the auxiliary equation distinct and real.

Lots of applications. E.g. restaurant swing door - choose mechanical parameters so that it is somewhere around critically damped ...
Overdamped, etc.

**Figure:** A solution of $y'' + cy' + y = 0$. (Has $y(0) = 0$, $y'(0) = 1$.) $c = 2$, critical damping

? Restaurant swing doors
Overdamped, etc.

**Figure:** A solution of $y'' + cy' + y = 0$. (Has $y(0) = 0$, $y'(0) = 1$.) $c > 0$ large (overdamped)
An easy numerical example, (N)

\[ y'' + 2y' + 5y = F \cos(\omega t) = F \text{Re} \left( \exp(i\omega t) \right), \quad (N) \]

a nonhomogeneous d.e., has real solutions \( y(t) = y_h(t) + y_p(t) \)

where

\[ y_p(t) = F \text{Re} \left( \frac{\exp(i\omega t)}{5 + 2i\omega - \omega^2} \right) \]

is a *particular solution* and \( y_h \) is the general solution of the homogeneous problem. \( \text{Re} \) means ‘real part’.

In other words, there are similarities with the situation with linear algebraic equations.
Digression on details

\[ y'' + cy' + y = F \cos(\omega t) = F \text{Re}(\exp(i\omega t)), \quad (N) \]

(Particular solution? Easiest way by hand: Substitute \( y_p = \text{Re}(A \exp(i\omega t)) \) into the d.e. and solve for the complex number \( A \).)

Will just talk about what you will see in the plots. Actually just a digression from my main point on linearity coming soon.
Digression - Resonance

Figure: A solution of \( y'' + cy' + y = \cos(wt) \). (Has \( y(0) = 0, y'(0) = 1 \).) \( c = 0, w = 1 \) is resonance
Figure: A solution of $y'' + cy' + y = \cos(wt)$. (Has $y(0) = 0$, $y'(0) = 1$.) $w = 1$, $c > 0$
Digression - Beats

Figure: A solution of $y'' + cy' + y = \cos(wt)$. (Has $y(0) = 0$, $y'(0) = 1$.) $c = 0$, $w = 15/16$, beats.

Piano tuner?
An easy numerical example, (N) - main point

A nonhomogeneous d.e. has solutions $y(t) = y_h(t) + y_p(t)$ where $y_p$ is a *particular solution* and $y_h$ is the general solution of the homogeneous problem. In other words, there are similarities with the situation with linear *algebraic* equations.
Review of 1st year linear algebra

- With $A$ a $m \times n$ matrix of real numbers, $0$ a $m \times 1$ (column) vector of zeros, the solution set of the homogeneous equation $Ax = 0$ is a vector space (actually a subspace of $\mathbb{R}^n$). (In first year, you learnt that this subspace is called the nullspace of $A$.)

- If you know one solution $x_p$ of the nonhomogeneous equation $Ax_p = b$, then any other solution is of the form $x_p + x_h$, where $x_h$ is a solution of the homogeneous equation $Ax_h = 0$.

- The equation $Ax = b$ has either no solution, precisely one solution or infinitely many solutions.
For the formal definition, see Wikipedia, if you want. We will just talk our way through the examples we will need.
\( \mathbb{R}^n \) example

\[
\mathbb{R}^n = \text{set of } n - \text{tuples of real numbers }, \\
= \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n \}.
\]

With obvious definitions of addition and of multiplication by a real number \( \mathbb{R}^n \) is a (real) vector space.

Perhaps the simplest instance of this is the case \( n = 2 \).

\[
\mathbb{R}^2 = \{ (x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R} \}
\]

The definition of addition is

\[(x_1, x_2) + (X_1, X_2) := (x_1 + X_1, x_2 + X_2)\]

and the definition of scalar multiplication is

\[\alpha(x_1, x_2) := (\alpha x_1, \alpha x_2) .\]
Consider continuous real-valued functions defined on a real interval 
$[a, b]$. Denote the set of all of these by $C([a, b])$. For $f, g$ in
$C([a, b])$, and $\lambda$ in $\mathbb{R}$, define $f + g$ and $\lambda f$ by

$$(f + g)(x) = f(x) + g(x),$$

$$(\lambda f)(x) = \lambda f(x).$$

Then $C([a, b])$ is a vector space.
Other vector spaces of functions, etc.

$C^k([a, b])$ real-valued functions on $[a, b]$ which are continuous and have $k$ derivatives, also continuous

$UP(x, n)$ univariate polynomials in a variable $x$ of degree $\leq n$.

$TP(x, n)$ truncated Fourier series - more on these later.

Spaces of sequences ...

All these have their uses, e.g. Fourier Series
Definition of ‘subspace’

Let $V$ be a vector space over a field $\mathbb{K}$. A subset $W$ of $V$ which is also a vector space over $\mathbb{K}$ is called a subspace of $V$. An equivalent characterisation is given in StrBC §2.1 p64. A set $S$ of vectors of $V$ is a subspace of $V$ if and only if

1. $\mathbf{0} \in S$;
2. (i) the sum of any two vectors in $S$ is in $S$, and (ii) the product of any scalar with any vector in $S$ is in $S$. (Actually (1) is a consequence of (2)(ii) with 0 as the scalar.)
Example

Let \( S \) be a subspace of \( \mathbb{R}^2 \). Then there are three possibilities:

0. \( S \) is the trivial subspace containing only the zero vector;

1. there is a straight line \( L \) through the origin such that \( S \) consists exactly of the vectors parallel to \( L \);

2. \( S = \mathbb{R}^2 \).
1.2 Subspaces

Definition. Let $V$ be a vector space over a field $K$. A subset $W$ of $V$ which is also a vector space over $K$ is called a subspace of $V$.

An equivalent characterisation is given in [StrBC] §2.1 p64j [StrWC] §3.1 2nd ed. p103, 3rd ed.

A set $S$ of vectors of $V$ is a subspace of $V$ if and only if

1. $0 \in S$;
2. (i) the sum of any two vectors in $S$ is in $S$, and (ii) the product of any scalar with any vector in $S$ is in $S$.

(Actually (1) is a consequence of (2)(ii) with 0 as the scalar.)

Some of your first year results are then stated in the following Easy (1st LA Course) Result and two Examples

1ST COURSE RESULT. Let $F$ and $G$ be nonzero vectors in $\mathbb{R}^2$ which are not parallel. Then every vector in $\mathbb{R}^2$ is the sum of a multiple of $F$ by a scalar and a multiple of $G$ by a scalar.

The first part of the next theorem is an immediate consequence of the preceding lemma.

EXAMPLE. Let $S$ be a subspace of $\mathbb{R}^2$. Then there are three possibilities:

1. $S$ is the trivial subspace containing only the zero vector;
2. there is a straight line $L$ through the origin such that $S$ consists exactly of the vectors parallel to $L$.

EXAMPLE. Let $S$ be a subspace of $\mathbb{R}^3$. Then there are four possibilities:

1. $S$ is the trivial subspace containing only the zero vector;
2. there is a straight line $L$ through the origin such that $S$ consists exactly of the vectors parallel to $L$;
3. there is a plane $\Pi$ through the origin such that $S$ consists exactly of the vectors parallel to $\Pi$.

1.3 Linear independence and dimension

[StrBC] §2.3j [StrWC] §3.1.

Definition. Let $V$ be a vector space over a field $K$. A linear combination of vectors $f_1, f_2, \ldots, f_k$ from $V$, is a sum $\sum_{i=1}^{k} c_i f_i$ in which each $c_i \in K$.

Figure: Subspaces of $\mathbb{R}^2$
The set of all solutions – as functions of $t$ on an interval $[a, b]$ – of a homogeneous (i.e. unforced) linear d.e. is a subspace of the vector space of all (appropriately) differentiable functions defined on $[a, b]$. 
1st year LA extends to vector spaces in general. In particular, one can define

- *linear combination of vectors*
- $\text{span}(S)$, the *span* of a set $S$ of vectors
- *linearly dependent, linearly independent, basis*
- *dimension*
**Definition.** A vector space with a finite basis is said to be *finite dimensional*; otherwise it is *infinite dimensional*.

After giving the definition of vector space we saw two examples: $\mathbb{R}^n$ which is finite dimensional, and the function space $C([a, b])$ which is infinite dimensional.
## Abstract setting | generalising | 1st course in LA

| (Abstract) vector spaces | generalise | $\mathbb{R}^n$ |
| Normed vector spaces | generalise | $\mathbb{R}^n$ with distances |
| Inner-product vector spaces | generalise | $\mathbb{R}^n$ with dot products |

You learnt in your 1st course in LA that dot products give you a method of treating angles between vectors. Inner product function spaces is the setting for Fourier Series, a topic treated later.
First results

\[(Ly)(t) := y'' + P_1(t)y' + P_0(t)y . \]

Defined in first year, linear 2nd order d.e.:

\[(Ly)(t) = f(t) \quad (N)\]

\[(Ly)(t) = 0 \quad (H)\]

(H) is a *homogeneous* d.e.: (N) stands for *non-homogeneous*.

**THEOREM.** The set of all solutions of a homogeneous linear d.e. forms a vector space.
From particular solution to general

**THEOREM.** Suppose that we have a particular solution $y_p$ of $Ly_p = f$. Then any (other) solution of $Ly = f$ is the sum of $y_p$ and a solution $y_h$ of the homogeneous d.e. $Ly = 0$.

These results are exactly like the corresponding results for algebraic equations given earlier. We stated them for 2nd order d.e.s, but they will work just as well for $n$-th order d.e.s for other values of $n$, including for $n = 1$. 

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ENVE3605 2nd order ODEs
A result which is much more difficult to prove (so proof is definitely not for this unit) is:

**THEOREM.** The space of all solutions of a homogeneous linear $n$th-order d.e. is $n$ dimensional.
The steps in solving linear d.e.s

- Find all solutions – the solution space – for the homogeneous d.e..
- Find a particular solution of the nonhomogeneous d.e..
- Put the preceding bits together.
Finding $y_p$? Variation of parameters

Once you know *all* the solutions of the corresponding homogeneous d.e., the ‘variation of parameters formula’ (which we will treat later) solves nonhomogeneous d.e.s.

This generalises what you did, in first year, for a single first order d.e.. (p13 Chpt1, notes.) Here is a reminder. Consider

\[
\frac{dy}{dt} + P(t)y = b(t). \tag{N}
\]

Suppose $\phi$ is a nonzero solution of

\[
\frac{d\phi}{dt}(t) + P(t)\phi(t) = 0 \quad \text{for all } t. \quad \phi(t_0) = 1 \tag{H}
\]

Then the solution of (N) with $y(t_0) = y_0$ is given by

\[
y(t) = \phi(t) \left( y_0 + \int_{t_0}^{t} \phi(s)^{-1} b(s) ds \right)
\]
Variation of parameters, ctd

This 1st order formula generalises! For now, it suffices to record:

SLOGAN. If you can find all the solutions of the homogeneous problem, there is a formula (‘variation of parameters’) for writing the solution of the nonhomogeneous problem in terms of integrals.

(CAUTION. Except if you are explicitly asked to do variation of parameters by hand, don’t do it. If you must do hand calculation, the ‘method of undetermined coefficients’ which you were shown in first year, when it is applicable, is easier. However, we recommend you use Matlab.)
Variation of parameters, single 2nd order d.e

p44 Chpt2. de: \( y'' + py' + qy = b \) \hspace{1cm} (N)

Let \( y_1 \) and \( y_2 \) be solutions of (H), de with \( b = 0 \).

To solve (N) seek a solution in form

\[ y = u_1y_1 + u_2y_2 \]

and try to determine the functions \( u_1 \) and \( u_2 \).

The determination of \( u_1 \) and \( u_2 \) proceeds as follows.

\[ y' = (u'_1y_1 + u'_2y_2) + (u_1y'_1 + u_2y'_2). \]

We are going to substitute the assumed form into d.e. (N). This will lead to one equation for the two unknowns \( u_1, u_2 \) so we are free to apply another condition, say

\[ u'_1y_1 + u'_2y_2 = 0 \] \hspace{1cm} (A)

Hence

\[ y' = u_1y'_1 + u_2y'_2 \]
Variation of parameters, single 2nd order d.e, ctd

Hence

\[ y' = u_1 y'_1 + u_2 y'_2 \]

and

\[ y'' = u_1 y''_1 + u'_1 y'_1 + u_2 y''_2 + u'_2 y'_2 \]

Next we put this into the d.e (N).
Variation of parameters, single 2nd order d.e, ctd

Next

\[ Ly = y'' + py' + qy \]
\[ = u_1(y_1'' + py_1' + qy_1) + u_1'y_1' + u_2(y_2'' + py_2' + qy_2') + u_2'y_2' \]
\[ = u_1'y_1' + u_2'y_2' = b \quad (B) \]

Equations (A)

\[ u_1'y_1 + u_2'y_2 = 0 \quad (A) \]

and (B) are linear algebraic equations for \( u_1' \) and \( u_2' \)

\[
\begin{bmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{bmatrix}
\begin{bmatrix}
u_1' \\
u_2'
\end{bmatrix}
=
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\]
Variation of parameters, single 2nd order d.e, ctd

Obviously can solve the pair of linear algebraic equations for $u_1'$ and $u_2'$.
Then integrate, then put it together to get the particular solution

$$y_p = u_1 y_1 + u_2 y_2$$

Although this is tolerably neat and memorable, and extends to single $n$-th order equations, all we need of you (in the exam, and in real life, I guess, though not necessarily in Assgt 1) is to know that “there is a formula”.


treats 2nd order, and $n$-th order equations.
Actually, I prefer, you to remember that the same general idea also works with ‘linear systems of 1st order equations’. More on this soon.
Variation of parameters, $y'' + y = b$

Write $c = \cos(t)$, $s = \sin(t)$, instead of the $y_1$ and $y_2$ we were using before as our solutions of the homogeneous d.e.

Our equations (A) and (B) are now

$$u_1' c + u_2' s = 0$$

$$u_1'(-s) + u_2' c = b$$

On using $c^2 + s^2 = 1$ we have

$$u_1' = -bs,$$

$$u_2' = bc$$

And, putting it all together

$$y_p = \cos(t) \int_0^t (-b(s)) \sin(s) \, ds + \sin(t) \int_0^t (b(s)) \cos(s) \, ds$$
\[ y'' + y = b, \ b \text{ constant} \]

Just a check. From before

\[ y_p = \cos(t) \int_0^t (-b(s)) \sin(s) \, ds + \sin(t) \int_0^t (b(s)) \cos(s) \, ds \]

With \( b(s) \) a constant, integrating gives

\[ y_p = b \cos(t)(\cos(t) - 1) + b(\sin(t))^2 = b(1 - \cos(t)) \]

Since the second term solves the homogeneous equation, equally good as a particular solution would be \( y_p = b = \text{constant} \).
(Of course this is obviously a particular solution, and we don’t need Variation of Parameters to tell us. The point here was to check that Variation of Parameters works.)
\[ y'' + y = b(t), \text{ other } b(t) \]

Easy. \( b(t) = t. \ y_p(t) = t \)

Bit harder, \( b(t) = \sin(t) \). Get resonance.

Q8(i) on Assgt 1 has \( b(t) = 1/\sin(t) \) and then the integrals are a bit nasty (and the current version of Matlab’s Symbolic Toolbox fails).
SLOGAN. If you can find all the solutions of the homogeneous problem, there is a formula (‘variation of parameters’) for writing the solution of the nonhomogeneous problem in terms of integrals.
Solving the homogeneous d.e.?

How can we find formulae to solve the homogeneous d.e.‘. For d.e.s of higher order than first, in fact, the answer is ‘you usual cannot’.

There are some special techniques of note


This is for the case where we have a formula for one of the solutions and it gives a technique whereby a second lin indep one can be found in terms of it and integrals.

(Additional examples of reduction of order are in the solutions to questions 6,7 and 9 of Assgt 1, but you are not asked to apply it yourself.)

However, there is an important special case when we can solve the homogeneous d.e.: the case of constant coefficients.

SLOGAN. Homogeneous constant coefficient d.e.s have (possibly complex) exponential solutions.
EXAMPLE
Repeated for the umpteenth time!
(A simple special case of what is in the notes)

\[ my'' + ky = 0 \]

Look for solutions \( y(t) = \exp(rt) \).

\[ mr^2 + k = 0 \]

Thus solutions are

\[ y(t) = C_1 \exp(i \sqrt{\frac{k}{m}} t) + C_2 \exp(-i \sqrt{\frac{k}{m}} t) \]

and the trivial matter of getting to real solutions was treated in first year.
Practical applications of 2nd order const coeff d.e.

See earlier ..., covering resonance and so on.
Noise measuring devices.
(Low frequency oscillations from ALCOA blasting - floppy membranes
See also the printed notes.
It was treated in 1st year. Revise it.
There are CAA questions on it.
Superposition is the key ingredient in determining the periodic response to periodic forcing for a (stable) constant coefficient (system of) d.e. From the Fourier Series of the input (forcing) it is easy to find the Fourier Series of the (long-time, periodic) output (response).

(I.e. the response after the transients have died away.)

There are CAA questions on this.
More.. superposition, Fourier Series

Superposition is the key ingredient in determining the periodic response to periodic forcing for a (stable) constant coefficient (system of) d.e. From the Fourier Series of the input (forcing) it is easy to find the Fourier Series of the (long-time, periodic) output (response).
(I.e. the response after the transients have died away.)
An example of hand calculation on this (of the kind you would have done in MATH2040 in 2nd year) is given on pp28-29 of the Fourier Methods notes.
More.. the general setting is systems

The single d.e.

$$y'' + P_1y' + P_0y = f$$  

(N)

can be written as a system of first order d.e.s by writing

$$u_1 = y, \quad u_2 = y'$$

so that

$$u_1' = u_2$$

$$u_2' = -P_0u_1 - P_1u_2 + f$$

More generally a single \( n \)-th order d.e. can be written as a system

$$u' = Au + f$$

where \( u \) is an \( n \)-vector with first coefficient \( u_1 = y \), etc. and \( A \) is an \( n \) by \( n \) matrix with entries functions of \( t \).