ENVE3605
2nd order PDE
Laplace equation

Grant Keady
Intro

Connection with holomorphic functions

Holomorphic functions, 2

exp(z)

z^2

Context

Porous Media

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Porous media flow through ‘triangular dam’

Flow through porous ‘triangular dam’, 2

Flow through porous ‘triangular dam’, 3

Well-set problems for elliptics

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A problem which is not well set

A typical boundary value problem
The Laplace equation for an unknown function of two variables $\varphi$ has the form

$$\varphi_{xx} + \varphi_{yy} = 0.$$ 

As in the last 2 lectures, elliptic PDE more generally share some properties - which problems are well set, etc.

Solutions of Laplace’s equation are called harmonic functions.
Connection with holomorphic functions

Solutions of the Laplace equation in two dimensions are intimately connected with analytic functions of a complex variable (a.k.a. holomorphic functions): the real and imaginary parts of any analytic function are conjugate harmonic functions: they both satisfy the Laplace equation, and their gradients are orthogonal. If \( f = u + iv \), then the Cauchy-Riemann equations state that

\[
\begin{align*}
    u_x &= v_y, \\
    v_x &= -u_y,
\end{align*}
\]

and it follows that

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \\
    v_{xx} + v_{yy} &= 0.
\end{align*}
\]

Conversely, given any harmonic function in two dimensions, it is the real part of an analytic function, at least locally.
## Holomorphic functions

<table>
<thead>
<tr>
<th>( f(z) )</th>
<th>( u = \text{Re}(f(z)) )</th>
<th>( v = \text{Im}(f(z)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exp(z) )</td>
<td>( \exp(x) \cos(y) )</td>
<td>( \exp(x) \sin(y) )</td>
</tr>
<tr>
<td>( z^2 )</td>
<td>( x^2 - y^2 )</td>
<td>( 2xy )</td>
</tr>
</tbody>
</table>

Note: both the \( u \) and the \( v \) are harmonic. The next displays give some pictures.
Figure: Curves correspond to $x = \text{const}$, and to $y = \text{const}$ for $\exp(x + iy)$
Figure: Curves correspond to $x = \text{const}$, and to $y = \text{const}$ for $z^2$. Streamlines and equipotentials for (inviscid) potential flow around a parabolic obstacle.
Context

There is more up at

http://en.wikipedia.org/wiki/Laplace_equation

There are Laplace-eq specific packages, e.g. in Matlab the Schwarz-Christoffel Toolbox.
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FLOW IN POROUS MEDIA

Gradients in head = h + gy

drive the flow.

Darcy’s law (1856) \[ \n = - k \nabla (h + gy) \quad (D) \]

where \( \n \) is ‘average’ velocity

\( k \) ‘permeability’

Conservation of mass

\[ \text{div} \ n = 0 \quad (C) \]

\( (C), (D) \Rightarrow \Delta h = 0 \)
B.C.s for FLOW IN POROUS MEDIA

At an impermeable boundary \( S(x,y)=0 \)
\[
\mathbf{u} \cdot \nabla S = 0
\]

At a free boundary \( F(x,y,t)=0 \)
\[
\begin{align*}
F_t + \mathbf{u} \cdot \nabla F &= 0 \\
\rho &= 0
\end{align*}
\]

Behrinder, Fluid Dynamics pp 223-224
Porous media flow, through ‘triangular dam’

Only here because the solution is nice and easy.
Flow through porous ‘triangular dam’, 2

Figure: Flow through porous ‘triangular dam’.
Impermeable base, no flow through it.
On CA: hydrostatic pressure in upstream (left) part
On AO: atmospheric pressure, water dribbles down to O
Water at zero depth rushes away on right hand side.
Flow through porous ‘triangular dam’, 3

Let $h_u$ denote the height of $A$ above the horiz. base. $O$ is at $(0,0)$, $C$ at $(-2h_u,0)$, $A$ at $(-h_u,h_u)$ $g = 1$ to save writing $g$ over and over.

The BV problem for the pressure $p$ is:

\[
\begin{align*}
    p &= h_u - y \quad \text{on } AC, \; x = -2h_u + y \\
    p_y &= -1 \quad \text{on } CO, \; y = 0 \\
    p &= 0 \quad \text{on } OA, \; x = -y
\end{align*}
\]
Well-set problems for elliptics

Require the elliptic pde to be solved with data given all the way around the boundary.
(2nd order) If the function is given, said to be Dirichlet data.
If the normal derivative is given, said to be Neumann data.
Our porous triangle has bits of both.

Solution \((h_u = 1)\)

\[
p = -x - y - \frac{1}{4}(x^2 - y^2)
\]

from which the last 2 BCs are easily checked.
Re-writes, e.g. to

\[
p + y - 1 = -\frac{1}{4}(2 + x - y)(2 + x + y)
\]

and the 1st BC checks.
Flow through porous ‘triangular dam’, 4

Figure: Flow through porous ‘triangular dam’. Contours of equal pressure

Ignore curves outside the triangle.
ezcontour likes rectangles.
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\[ \phi = k(p+gy); \quad u = (u,v) \]

\[ u = \phi_x \quad v = \phi_y \quad \phi = -(p+gy) \]

\( \psi \) is harmonic conjugate to \( \phi \):

\[ w = u - iv \quad \text{complex velocity} \]

\[ \chi = \phi + i\psi \quad z = x + iy \]

\[ u - iv = w = \frac{dx}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \]

\[ = -i \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} \]

\text{Cauchy-Riemann}

\[ u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \]

\[ v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \]
Flow through porous ‘triangular dam’, 5

Figure: Flow through porous ‘triangular dam’.
Streamlines. $\psi = y(1 + x/2)$

Ignore curves outside the triangle.
A problem which is not well set

In our first lecture this week, on classification of PDE, we saw that the Cauchy IV Problem for the Laplace equation was not well set.

Well-set problems for the Laplace equation involve having boundary data specified all the way around the boundary. Our porous triangle problem is an example.
A typical boundary value problem

A typical problem for Laplace’s equation is to find a solution that takes prescribed values on the boundary of a domain. For example, we may seek a harmonic function that takes on the values $u(\theta)$ on a circle of radius one. The solution was given by Poisson:

$$\varphi(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \theta')} u(\theta') \, d\theta'.$$

This formula can be obtained by summing a Fourier series for $\varphi$. If $r < 1$, the derivatives of $\varphi$ may be computed by differentiating under the integral sign, and one can verify that $\varphi$ is analytic, even if the boundary data is continuous but not necessarily differentiable. This behavior is typical for solutions of elliptic partial differential equations: the solutions may be much more smooth than the boundary data. This is in contrast to solutions of the wave equation, and more general hyperbolic partial differential equations, which typically have no more derivatives than the data.
Using Fourier series for some problems for the Laplace Equation

The HELM notes, Chpt 25, §25.4 has several examples, worked in (sometime tedious) detail, with hand-calculation of the Fourier Series and all.

Assgt2 Q6(i) has another example.

Here we present just one example.
PROBLEM. Solve Laplace’s Equation in the semi-infinite strip \( \{(x, y) | -\pi < x < \pi, y > 0\} \) with the solution \( u \) (e.g. temperature) prescribed around the boundary as follows:

\[
\begin{align*}
    u(-\pi, y) &= 0 = u(\pi, y) \text{ for } y > 0, \\
    u(x, 0) &= \pi - |x| \text{ for } -\pi < x < \pi.
\end{align*}
\]

We require a solution which tends to 0 as \( y \) tends to plus infinity. 

Solution. We begin by noting that the domain and the boundary conditions are symmetric in \( x \) so Fourier cosine series will be appropriate. The functions \( \cos((n + 1/2)x) \) vanish at \( x = \pm \pi \). When \( n \) is a non-negative integer, the functions \( \exp(-(n + 1/2)y) \cos((n + 1/2)x) \) satisfy the Laplace equation in the semi-infinite strip and the zero b.c.s on \( x = \pm \pi \),
Fourier Series for Laplace Eq in a strip, 2

We can seek a solution to our problem in the form of a Fourier cosine series

\[ u(x, y) = \sum_{n=0}^{\infty} A_n \exp\left(-\left(n + \frac{1}{2}\right)y\right) \cos\left((n + \frac{1}{2})x\right). \]

All that remains to be done is to find the \( A_n \) so that the boundary condition at \( y = 0 \) is satisfied.

The Fourier cosine series for a triangular wave is found in the usual way.

We have the orthogonality relation

\[ \int_{-\pi}^{\pi} \cos((m + 1/2)x) \cos((n + 1/2)x) \, dx = \pi \delta(m, n) \]

where \( \delta(m, n) = 0 \) for \( m \neq n \) and \( \delta(m, m) = 1 \).
The table on p18 of the ‘Fourier Methods’ handout (a table which was also in the MATH2040 notes) treats a triangular wave and we could use that.

Here, however, we begin with noting that

\[
A_n = \frac{2}{\pi} \int_0^\pi \cos((n + 1/2)x)(\pi - x) \, dx
\]

\[
= \frac{8}{(2n + 1)^2\pi}
\]

Putting it all together

\[
u(x, y) = \frac{8}{\pi} \sum_{n=0}^{\infty} \exp(-(n + 1/2)y) \frac{\cos((n + 1/2)x)}{(2n + 1)^2}
\]
Figure: Solution of Laplace’s equation in a semi-infinite strip. BCs as described in the text.
Fourier transforms

Appropriate for half-space domains, e.g. \( \{ y > 0, \text{ all } x \} \).

See Assgt 2 Q5.
Another analytical method is ‘Schwarz Christoffel’. Restriction: must be dealing with polygonal domains.

See Matlab SchwarzChristoffel Toolbox. (free, open source)
More dimensions, e.g. 3D

Fourier methods carry over.
Complex variable methods do not.
Further elliptic pde problems

The steady states that are reached in any of the parabolic pde applications mentioned in our lecture on the heat eq and parabolics satisfy elliptic pde BV problems.
Further analytical methods - maximum principles

The Maximum Principle says for the (homogeneous) Laplace eq. (and eqs like it) that the maximum temperature (of a nonconstant solution) cannot occur inside the domain. The max must occur on the boundary.

Can be used to prove uniqueness of solutions. Also ‘comparison methods’.

In any event, bounds on quantities (not just the function $u$ but also the size of the gradient) are of use.

You don’t have to get a formulae for the solution to understand what the solutions are like ... (though it is nice when one can, even if it only gets used to check numerical methods).