MATH3601
FOURIER METHODS

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The Fourier Series content of these notes is revision of 2nd year.
The Fourier Transform and fft Chapters are new to most of you in MATH3601.
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Chapter 1

INTRODUCTION TO FOURIER SERIES

Fourier methods are used to analyse wave forms (or signals) giving a breakdown of the wave form into its various frequency components. Amongst the many uses we mention the electronic, mechatronics, acoustic and communication engineering applications where people wish to know what effect filters have on signals with a particular frequency. For example, it may be desired to filter out any unwanted frequency components from a signal. By studying the effect of a filter on the various Fourier components, we can understand the effect of the filter on the original signal itself.

There are other uses – but the preceding paragraph will suffice for now.

1.1 Two approaches to teaching Fourier Series

There are really two approaches to how to begin a treatment of Fourier Series.

• One way, and the supplement on Fourier Series which is produced to go with Stewart ‘Calculus’ is representative of this, assumes no knowledge of linear algebra or matrix algebra. Fourier – who was friend of Napoleon, which places the period in which he worked – didn’t know about inner product spaces, but it didn’t stop him doing useful work with Fourier series.

• The other way, as for example in Chapter 3 (on orthogonality) of Strang’s book, [StrBC], is to start with inner product spaces and orthogonality.

You should study, in your own time, the Stewart supplement which uses the calculus-only approach. These notes begin with the inner-product-space orthogonality approach. Because we also want to treat applications of Fourier series to d.e.s we also, in Chapter 2, give more examples of Fourier Series, and here the calculations are exactly like those of the Stewart supplement.

1.2 Abstract inner product spaces

Really the situation is just like \( \mathbb{R}^n \) with dot products and the standard basis \( \{e_j\}_{j=1}^n \). Denote the inner product with the angle-brackets, \( \langle u, v \rangle \): it is just the dot product \( u \cdot v \). Given a vector \( f \) in
this inner-product space, we have (and it is easy...)

\[ f = \sum_{j=1}^{n} \langle f, e_j \rangle e_j. \]

Actually there would be a similar formula for any orthonormal basis.

So far this is just an example of orthogonal bases in an example of an inner-product space. To make progress we need an abstract treatment. Let’s get down to doing this. Let \( V \) be a real inner product space, and denote its inner-product with the angle-brackets as usual. Let \( \{ \phi_j \}_{j=0}^{\dim(V)} \) be an orthogonal basis for \( V \). Then, for \( f \in V \),

\[ f = \sum_{j=0}^{\dim(V)} c_j \phi_j \quad \text{with} \quad c_j = \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}. \]

(In a complex inner product space, some complex conjugates would appear in some formulae.) The Main Fact of Fourier Series is really just this but applied to an inner-product function space which we will describe in the next section, and with an orthogonal basis consisting of a collection of sine and cosine functions.

### 1.3 Fourier series and the inner product space of periodic square-integrable functions

A periodic function \( f(x) \) is a function which is defined for all real values of \( x \) and for which there exists a positive number \( p \) so that

\[ f(x + p) = f(x). \]

The number \( p \) is called the **period** of \( f(x) \). (The set of all functions with given period \( p \) is a vector space.) Here follow some examples of periodic functions:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos(x) )</td>
<td>( 2\pi )</td>
</tr>
<tr>
<td>( \sin(x) )</td>
<td>( 2\pi )</td>
</tr>
<tr>
<td>( \exp(\sin(\sqrt{2}x)) )</td>
<td>( \sqrt{2}\pi )</td>
</tr>
<tr>
<td>( \cos(2x) )</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

Let us consider the inner-product space of real-valued functions which are of period \( 2L \) for some number \( L \), and use as inner product

\[ \langle f, g \rangle = \int_{-L}^{L} f(x)g(x) \, dx. \]

The set of all functions \( f \) such that \( \langle f, f \rangle \) is defined is called the **inner-product space of square-integrable functions** and denoted \( L_2 \) (or, if we need to remind ourselves of the period, by \( L_2(-L, L) \) or by \( L_{2,\text{per}}(-L, L) \) or something like this). Now define the functions

\[ c_n(x) = \cos\left( \frac{n\pi x}{L} \right) \quad \text{for} \quad n = 0, 1, 2, \ldots, \]

\[ s_n(x) = \sin\left( \frac{n\pi x}{L} \right) \quad \text{for} \quad n = 1, 2, \ldots. \]
1.3. FOURIER SERIES AND THE INNER PRODUCT SPACE OF PERIODIC SQUARE-INTEGRABLE FUNCTIONS

For any positive integer \( m \), define

\[
\mathcal{B}(m) = \{1, c_1, s_1, c_2, s_2, \ldots, c_m, s_m\}.
\]

\( \mathcal{B}(m) \) is an orthogonal set of functions. So far this easy, but we also want to let \( m \) be \( \infty \). So also let

\[
\mathcal{B} = \bigcup_{m=0}^{\infty} \mathcal{B}(m)
\]

Again, the functions in \( \mathcal{B} \) are pairwise orthogonal.

**MAIN FACT.** \( \mathcal{B} \) is an orthogonal basis for the inner-product space \( L_2 \) of square integrable functions of period \( 2L \).

**RESTATEMENT OF THE ABOVE MAIN FACT.** Fourier’s insight was that any \( 2L \)-periodic square-integrable function \( f \) (pretty well anything that happens in engineering applications with periodic functions) can, by taking \( m \) sufficiently large, be represented arbitrarily closely by taking the sensible linear combination of the functions in \( \mathcal{B}(m) \):

\[
f(x) = a_0 + \sum_{j=1}^{m} (a_j c_j(x) + b_j s_j(x)).
\]

where

\[
a_0 = \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle}, \quad a_j = \frac{\langle c_j, f \rangle}{\langle c_j, c_j \rangle}, \quad b_j = \frac{\langle s_j, f \rangle}{\langle s_j, s_j \rangle}.
\]

There are a lot of technical issues associated with letting the number of terms, or equivalently \( m \), tend to infinity. Though it is a lot of work, it can be proven (and the proofs will not be in MATH3601, see later in these notes) that for a wide class of periodic functions, its Fourier series converges to the function itself.

In this unit, we will not treat in any serious way the underlying convergence theory, but will concentrate on some examples of how a Fourier series approximates a given function and uses of Fourier series for linear d.e. problems. We will also look at Fourier series of even and odd functions and give a complex-exponential treatment for Fourier series.

1.3.1 The easy integrals to establish the orthogonality

You don’t want to have to repeat these integrals every time you evaluate a Fourier series, of course, but at some stage (here!) you should see that the orthogonality relations are easy to establish. Of course \( \langle s_m, c_n \rangle = 0 \) is easy because the integral of an odd function over a symmetric range is zero. We will next show \( \langle c_m, c_n \rangle = 0 \) (and it happens that showing \( \langle s_m, s_n \rangle = 0 \) is very similar).

The starting point is school level trigonometry:

\[
\cos(A) \cos(B) = \frac{1}{2} (\cos(A - B) + \cos(A + B)).
\]

Then

\[
\langle c_m, c_n \rangle = \int_{-L}^{L} \cos\left(\frac{n \pi x}{L}\right) \cos\left(\frac{m \pi x}{L}\right) dx
\]

\[
= \frac{1}{2} \left( \int_{-L}^{L} \cos\left(\frac{(n-m) \pi x}{L}\right) dx + \int_{-L}^{L} \cos\left(\frac{(n+m) \pi x}{L}\right) dx \right)
\]

\[
= \frac{L}{2\pi} \left( \frac{\sin\left(\frac{(n-m) \pi x}{L}\right)}{n-m} \bigg|_{-L}^{L} + \frac{\sin\left(\frac{(n+m) \pi x}{L}\right)}{n+m} \bigg|_{-L}^{L} \right)
\]

\[
= 0,
\]

where for the last step we have used the fact that \( n, m \) are distinct nonnegative integers.
1.4 Definition of a Fourier series

We now just set \( m \), in the basis set \( \mathcal{B}(m) \) defined before, to be infinity. Let \( f \) be a periodic function with period \( 2L \). We define the Fourier series of \( f \) by the expression at the right of the equation below.

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right),
\]

with

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx, \quad n = 1, 2, \ldots,
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx, \quad n = 1, 2, \ldots.
\]

This is as in the Stewart Supplement, box 9, page 6. For most functions arising in engineering and continuous on the whole real line, the sum of the Fourier series will be identical to the original function, i.e. the Fourier series does converge to the function. These formulae for \( a_n \) and \( b_n \) are identical to those given in the inner-product space introduction above. We have written them in full to emphasise that it is just a matter of evaluating some integrals to find the Fourier series representation of function \( f \).

There are many variants of the formulae. The formulae given represent the Fourier series of a periodic function with a full cycle over the interval \( -L \leq x \leq L \). If the full cycle considered is over the interval \( 0 \leq x \leq 2L \), the integrals in (1.2) – (1.4) need to be evaluated between \( x = 0 \) and \( x = 2L \).

1.5 Fourier series and least squares approximation

This is related to understanding why Fourier series should converge (in a ‘least-squares’ sense, anyway: see later in these notes) to the given function.

Return to the notation we had before: \( \mathcal{B}(m) \) denotes the \((2m+1)\) functions we had abbreviated down to \( c_n, s_n \). One can consider the span of the set \( \mathcal{B}(m) \). This span is a \((2m+1)\) dimensional subspace of the infinite dimensional space of \( 2L \)-periodic functions. It turns out (and it is not all that hard to show) that the ‘least-square approximation’ (best approximation) to a given function \( f \) in \( \text{span}(\mathcal{B}_m) \) is given by the Fourier approximation (the Fourier series but stopping at \( m \)). Let \( S_m \) denote this partial Fourier sum in \( \text{span}(\mathcal{B}_m) \). See Figure 1.1 for a picture.

What the Fourier coefficients do is this. Let \( \| u \|^2 = \langle u, u \rangle \). Then the truncated partial Fourier sum \( s_m \) satisfies

\[
\| S_m - f \|^2 = \min_{f_m \in \text{span}(\mathcal{B}_m)} \| f_m - f \|^2.
\]

In the picture, think of the whole of the plane as representing the infinite dimensional vector space \( L_2 \) of all the \( 2L \)-periodic functions, and the line as representing the \((2m+1)\) dimensional subspace \( W_m = \text{span}(\mathcal{B}_m) \). We also write

\[
S_m = \text{proj}_{W_m}(f)
\]

and say that the projection of \( f \) on \( W_m \) is \( S_m \).
1.6. A COMMENT ON HOW TO READ THESE NOTES

Anyway, the story is that we are doing as well as we could do by using the Fourier coefficients.

Here is an item of caution about the notation. The use of \( s_n(x) = \sin(\pi x / L) \) was a local notation that won’t be used often. The letter \( s \) is also a good letter for sums and partial-sums, so I may occasionally write \( s_m \) where I have written \( S_m \) above.

As an aside, you should know that there are lots of other applications of ‘least squares’. Whenever you have lots of linear equations and only a few unknowns \( x \) to find, \( Ax = b \), the idea is that usually you won’t be able to solve them exactly. However, it may be that the appropriate approximation is the one which, using now the dot product in \( \mathbb{R}^n \) rather than our inner product in a function space, is that which makes \( \| Ax - b \|^2 \) as small as possible when you vary over \( x \).

We treated the finite-dimensional, matrix, version of least squares earlier in the set of notes (after our treatment of orthogonal matrices, projection matrices, positive definite matrices, etc.). The problem happens lots in engineering, and, in fact if you enter to Matlab \( A \backslash b \), the same syntax as solving when \( A \) is square, when the matrix is not square, it finds the least squares solution.

1.6 A comment on how to read these notes

These notes – especially the chapters on calculating Fourier series and on d.e. applications – are made rather long because they have been deliberately written with the hand calculations shown in full. In practice, you will often do you calculations aided with appropriate software – Matlab with its Symbolic Toolbox. The software makes the routine tasks like evaluating integrals easy – often as easy as typing one line of code. However, we thought about how you might read this, and it may well be away from a computer. It is important, though, NOT to get distracted by long hand calculations. You should spend some time, probably together with another person or two in this class, working the steps of a few of the problems worked in these notes with Matlab helping you with some of the steps. Again, you should avoid wasting time. There will be occasions when different equivalent forms of results are possible. Sometimes asking the software to transform between them is harder than inventing other checks, e.g. looking at representative numerical values and plots and so on. So do not become anxious about the different sorts of computer simplifications of expressions: if it helps to switch to hand calculation every so often, do so.

1.7 Some examples

Clearly the recipe we have just described turns the process of finding Fourier series into a sequence of routine steps. Although you should do at least one example by hand – presumably a fairly simple
one – you may find it less tiresome if you use Matlab’s Symbolic Toolbox to automate some of the calculations. I recommend that you store the following in files afourier.m and bfourier.m. These are presented in the form appropriate to $2L$-periodic functions defined on $(-L, L)$, but you could alter the code to cover the case of formulae defining the function on $(0, 2L)$ if you need to do so.

function a=afourier(n)
% afourier.m computes cosine coeffs in Fourier Series
    global L f x
    syms L f x pisym fl
    pisym = sym(pi);
    L=pisym; f = x; % sawtooth
    fl = f*cos(n*pisym*x/L);
    a = int( fl, x, -L, L )/L;

function b=bfourier(n)
% bfourier.m computes sine coeffs in Fourier Series
    global L f x
    syms L f x pisym fl
    pisym = sym(pi);
    L=pisym; f = x; % sawtooth
    fl = f*sin(n*pi*x/L);
    b = int( fl, x, -L, L )/L;

Next in a Matlab session define the period and the function, e.g. for the sawtooth waveform example below:

global L f x
syms L f x
L=sym(pi);
f = x; % sawtooth

% Then, for example, find the coefficients a0, a1, and b1.
% Recall the factor of 2 in definition of a0.
a0 = afourier(0)/2
a1 = afourier(1)
% For an odd function all the a are 0
b1 = bfourier(1)
b2 = bfourier(2)
b3 = bfourier(3)
b4 = bfourier(4)
b5 = bfourier(5)
% spot the pattern? bn=(2*(-1)^(n+1))/n
1.7. SOME EXAMPLES

1.7.1 A periodic function with period \(2\pi\)

Let us calculate the Fourier series of the function

\[ f(x) = x^2 \quad \text{if} \quad 0 < x < 2\pi, \]  

(1.5)

with period \(p = 2\pi\) \((L = \pi)\). The coefficients of the Fourier series are given by (try to verify these results yourselves, using Matlab’s Symbolic Toolbox to perform the integrals, or even the code provided above which does this – calls \texttt{int} to do the integrations – , if you wish):

\[
\begin{align*}
a_0 &= \frac{1}{2\pi} \int_0^{2\pi} x^2 \, dx \\
&= \frac{4\pi^2}{3}, \\
a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(nx) \, dx \\
&= \frac{4}{n^2}, \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin(nx) \, dx \\
&= -\frac{4\pi}{n}.
\end{align*}
\]

The Fourier series can then be written as

\[
f(x) = \frac{4\pi^2}{3} + \left( 4 \cos(x) + \cos(2x) + \frac{4}{9} \cos(3x) + \ldots \right) - \pi \left( 4 \sin(x) + 2 \sin(2x) + \frac{4}{3} \sin(3x) + \ldots \right).
\]  

(1.6)
Figure 1.3: The sum of the first five terms in the Fourier series of the periodic square wave.

For this function $f(x)$, we plot in Fig. 1.2, the approximation obtained by truncating the Fourier series after the third term in the sine and cosine. This means that terms with a period of $2\pi/3$ (or a frequency three times the basic frequency) are included.

1.7.2 The periodic square wave (even in $x$)

The Fourier series of the function

$$f(x) = \begin{cases} 
0 & \text{if } -2 < x < -1, \\
k & \text{if } -1 < x < 1, \\
0 & \text{if } 1 < x < 2,
\end{cases}$$

with period $p = 4$ ($L = 2$) has the following coefficients:

$$a_0 = \frac{1}{4} \int_{-2}^{2} f(x) dx = \frac{1}{4} \int_{-1}^{1} k dx,$$

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \left( \frac{n\pi x}{2} \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} k \cos \left( \frac{n\pi x}{2} \right) dx,$$

$$b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin \left( \frac{n\pi x}{2} \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} k \sin \left( \frac{n\pi x}{2} \right) dx.$$

These coefficients evaluate as

$$a_0 = \frac{k}{2},$$
1.8. EVEN AND ODD FUNCTIONS

\[ a_n = \begin{cases} 0 & \text{for } n \text{ even,} \\ \frac{2k}{n\pi} & \text{for } n = 1, 5, 9, \ldots, \\ -\frac{2k}{n\pi} & \text{for } n = 3, 7, 11, \ldots, \\ b_n = 0 & \text{for all } n. \end{cases} \]

The Fourier series itself can then be written as

\[ f(x) = k \frac{2}{\pi} + \frac{2k}{\pi} \left( \cos \left( \frac{\pi x}{2} \right) - \frac{1}{3} \cos \left( \frac{3\pi x}{2} \right) + \frac{1}{5} \cos \left( \frac{5\pi x}{2} \right) - \ldots \right). \]

See Fig. 1.3 for a plot of the approximation obtained by truncating the Fourier series after the fifth term.

Notice that only cosine terms appear in the Fourier series. This is because \( f(x) \) is an even function. For even functions, the coefficients \( b_n \) of the sine terms are all zero!

1.7.3 The sawtooth waveform (odd in \( x \))

The function

\[ f(x) = x \quad \text{if} \quad -\pi < x < \pi, \]

with period \( p = 2\pi \) \((L = \pi)\), has a Fourier series with the following coefficients:

\[ a_0 = 0, \]
\[ a_n = 0 \quad \text{for all } n, \]
\[ b_n = (-1)^{n+1} \frac{2}{n}. \]

The resulting Fourier series is given by

\[ f(x) = 2 \left( \sin(x) - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \frac{1}{5} \sin(5x) - \ldots \right). \]

See Fig. 1.4 for a plot of the approximation obtained by truncating the Fourier series after the fifth term.

The sawtooth waveform has only sine terms in its Fourier series. This is because \( f(x) \) is an odd function. For odd functions, the coefficients \( a_n \) of the cosine terms are all zero!

1.7.4 Matlab graphics

The plots above were produced with Matlab. The book Harman, Dabney and Richert Advanced Engineering Mathematics with Matlab (2nd ed.) provides an m-file (Example 8.2, page 374) to do this. Listen in lectures for the address from which it can be downloaded.

1.8 Even and odd functions

The last two examples show that the Fourier series simplify considerably for even or odd functions. The reason for this is that \( \cos(n \pi x/L) \) is an even function while \( \sin(n \pi x/L) \) is an odd function. Moreover, the product of odd and even functions behaves as given in the table below:
In addition, the typical integrals appearing in the formula for the Fourier coefficients can be simplified for even and odd functions:

\[ \int_{-L}^{L} g(x) \, dx = 2 \int_{0}^{L} g(x) \, dx \text{ for } g(x) \text{ even}, \]
\[ \int_{-L}^{L} g(x) \, dx = 0 \text{ for } g(x) \text{ odd}. \]

Using these properties one can easily show that the Fourier series for an even function reduces to

\[ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right), \quad (1.7) \]
\[ a_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx, \quad (1.8) \]
\[ a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx, \quad n = 1, 2, \ldots. \quad (1.9) \]

For odd functions the Fourier series can be written as

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right), \quad (1.10) \]
\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \quad n = 1, 2, \ldots. \quad (1.11) \]

If you know that your function is odd (say, or it is as good to know it is even) you can save a lot of time by using the information.
1.8.1 An example of orthogonal subspaces

Let \( X_e \) be the 2\( L \)-periodic functions which are even.
Let \( X_o \) be the 2\( L \)-periodic functions which are odd.
Clearly, for any \( f_e \in X_e \) and \( f_o \in X_o \),
\[
\langle f_e, f_o \rangle = \int_{-L}^{L} f_e(x)f_o(x) \, dx = 0 ,
\]
as we are integrating an odd function over a symmetric range. More generally two subspaces are said to be orthogonal subspaces if everything in one is orthogonal to everything in the other. The intersection of a pair of orthogonal subspaces is the zero vector.
In the special case of \( X_e, X_o \) we have that every 2\( L \)-periodic function \( f \) can be written as the sum \( f = f_e + f_o \). Here
\[
f_e(x) = \frac{1}{2} (f(x) + f(-x)) , \quad f_o(x) = \frac{1}{2} (f(x) - f(-x)) .
\]
\( \mathcal{B}_o = \{ \sin(n\pi x/L) \}_{n=1}^{\infty} \) is a basis for the subspace of odd functions in the inner-product space \( L_2(-L, L) \).
\( \mathcal{B}_e = \{ \cos(n\pi x/L) \}_{n=0}^{\infty} \) is a basis for the subspace of even functions in the inner-product space \( L_2(-L, L) \).

1.8.2 Derivatives of odd and of even functions

The derivative of a (differentiable) even function is odd.
The derivative of a (differentiable) odd function is even.

1.9 Half-Range expansions

Often a function is only defined on a particular interval \( 0 \leq x \leq L \). In some applications it is known that it is appropriate to extend the function to be odd and period defined for \(-\infty \leq x \leq \infty \). In other applications, it is much the same except the function is extended to be even and periodic.
You may well see these sorts of things in connection with other engineering units. These sorts of extensions typically occur with 2-point boundary-value problems for some d.e.s, and with these arising from ‘separation of variables’ solutions of some partial differential equation problems. (You might look at the Matlab lab on the solution of the wave equation. This is optional for students in 2nd year but not for students in MAEE.)
Let us consider the function
\[
f(x) = x^2 \text{ if } 0 \leq x \leq 2\pi .
\]

1.9.1 Even extension

In some circumstances it may be appropriate to extend \( f(x) \) as
\[
g(x) = x^2 \text{ for } -2\pi \leq x \leq 2\pi ,
\]
Figure 1.5: The sum of the first five terms in the Fourier series of
(i) the even extension and
(ii) the odd extension
of \( f(x) = x^2 \) on \((0, \pi)\).

with a period \( P = 4\pi \) \((L = 2\pi)\). The Fourier series then becomes a Fourier cosine series with as coefficients

\[
a_0 = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} x^2 \, dx = \frac{4\pi^2}{3},
\]

\[
a_n = \frac{1}{2\pi} \int_{-2\pi}^{0} x^2 \cos \left( \frac{nx}{2} \right) \, dx = (-1)^n \frac{16}{n^2}.
\]

This gives the following Fourier cosine series:

\[
g(x) = \frac{4\pi^2}{3} - 16\pi \cos \left( \frac{x}{2} \right) + 4 \cos(x) - \frac{16}{9} \cos \left( \frac{3x}{2} \right) + \ldots
\]

1.9.2 Odd extension

If we extend \( f(x) \) as an odd function \( g(x) \),

\[
g(x) = \begin{cases} 
-x^2 & \text{for } -2\pi \leq x \leq 0, \\
x^2 & \text{for } 0 \leq x \leq 2\pi,
\end{cases}
\]

the Fourier series becomes a Fourier sine series with as coefficients

\[
b_n = \frac{1}{2\pi} \left( \int_{-2\pi}^{0} -x^2 \sin \left( \frac{nx}{2} \right) \, dx + \int_{0}^{2\pi} x^2 \sin \left( \frac{nx}{2} \right) \, dx \right)
\]
Thus the Fourier sine series can be written as
\[
g(x) = \left(8\pi - \frac{32}{\pi}\right) \sin\left(\frac{x}{2}\right) - 4\pi \sin(x) + \left(8\pi - \frac{32}{27\pi}\right) \sin\left(\frac{3x}{2}\right) - \ldots
\]

Fig 1.5 pictures these different Fourier series approximations, each one truncated after the fifth term, i.e. five coefficients need to be calculated for each series. Please note that you can only rely on the Fourier series approximation to agree with the simple formula (here \(f(x) = x^2\) in the domain of the function \(f(x)\) (here \(0 \leq x \leq 2\pi\)). In this domain, the different Fourier series approximate the same curve, but outside this domain, of course, they differ considerably.
1.10 A short table of Fourier series
1.10. A SHORT TABLE OF FOURIER SERIES
1.11 A comment on pointwise convergence

For all reasonable functions occurring in engineering, the Fourier series of \( f \) converges at points of continuity to \( f(x) \) and at points where there is a jump discontinuity to the value exactly in the middle of the jump, \( (f(x^+) + f(x^-))/2 \). See the Stewart Supplement box 8 page 5, and also later in these notes.

1.12 Complex Fourier series

It is sometimes easier to work with the complex form of the Fourier series. The Euler formula
\[
e^{it} = \cos(t) + i \sin(t),
\]
can be applied for \( t = n\pi x/L \) and \( t = -n\pi x/L \),
\[
e^{i(n\pi x/L)} = \cos \left( \frac{n\pi x}{L} \right) + i \sin \left( \frac{n\pi x}{L} \right),
\]
\[
e^{-i(n\pi x/L)} = \cos \left( \frac{n\pi x}{L} \right) - i \sin \left( \frac{n\pi x}{L} \right).
\]

We can combine these two equations to obtain expressions for the sine and cosine terms:
\[
\cos \left( \frac{n\pi x}{L} \right) = \frac{1}{2} \left( e^{i(n\pi x/L)} + e^{-i(n\pi x/L)} \right),
\]
\[
\sin \left( \frac{n\pi x}{L} \right) = \frac{1}{2i} \left( e^{i(n\pi x/L)} - e^{-i(n\pi x/L)} \right).
\]

When these expressions for cosine and sine are substituted into the expression for the Fourier series (1.1), we obtain the complex form of the Fourier series
\[
f(x) = c_0 + \sum_{n=1}^{\infty} \left( c_n e^{i(n\pi x/L)} + k_n e^{-i(n\pi x/L)} \right),
\]
with the coefficients being defined by
\[
c_n = \frac{1}{2} \left( a_n - ib_n \right) = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i(n\pi x/L)} dx,
\]
\[
k_n = \frac{1}{2} \left( a_n + ib_n \right) = \frac{1}{2L} \int_{-L}^{L} f(x) e^{i(n\pi x/L)} dx = c_{-n}.
\]

Therefore, we can write the complex Fourier series more simply,
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i(n\pi x/L)},
\]
with the coefficient \( c_n \) given by
\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i(n\pi x/L)} dx.
\]
As an example, consider the function
\[ f(x) = e^x \text{ for } -\pi \leq x \leq \pi, \]
with period \( P = 2\pi \) \((L = \pi)\). The coefficient \( c_n \) can be easily then calculated as
\[

c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} \, dx \\
&= \frac{1}{2\pi} \frac{1}{1-in} \left( e^{(1-in)\pi} - e^{(1-in)(-\pi)} \right).
\]

When we use the identities
\[ e^{in\pi} = e^{-in\pi} = (-1)^n, \]
the coefficient can further be simplified as
\[

c_n &= \frac{1}{2\pi} \frac{1 + in}{1 + n^2} (-1)^n \left( e^\pi - e^{-\pi} \right) \\
&= \frac{\sinh(\pi) \left( 1 + \frac{in}{1 + n^2} \right)}{\pi} (-1)^n.
\]
The complex Fourier series can then be written as
\[

f(x) = \frac{\sinh(\pi)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1 + in}{1 + n^2} e^{inx}.
\]

If we write this series out in terms of sine and cosine, making use of symmetries between terms with opposite \( n \)'s, we find the real Fourier series as
\[
f(x) = \frac{2\sinh(\pi)}{\pi} \left( \frac{1}{2} - \frac{1}{2}(\cos(x) - \sin(x)) + \frac{1}{5}(\cos(2x) - 2\sin(2x)) - \ldots \right).
\]

### 1.13 Differentiation of Fourier series

If a function is equal to the sum of a finite number of terms of a Fourier series
\[ \phi(x) = a_0 + \sum_{n=1}^{N} (a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right)) \]
then, clearly its derivative is
\[ \phi'(x) = \sum_{n=1}^{N} (-a_n n \sin\left(\frac{n\pi x}{L}\right) + b_n n \cos\left(\frac{n\pi x}{L}\right)). \]

In other words, differentiating the series termwise gives a series which sums to the derivative \( \phi'(x) \).

When there are infinitely many nonzero terms in the Fourier series, more care is needed. Here are the relevant facts.

1. If the periodic function is very smooth on the whole real line, its Fourier coefficients tend to zero rapidly.

2. If the Fourier coefficients tend to zero sufficiently rapidly, the series (converging to \( f(x) \)) can be differentiated termwise (and the series so formed converges to \( f'(x) \)).

Of course, the finite sum at the beginning of this section is a special case. All the \( a_n \) and \( b_n \) for \( n > N \) are zero, so very rapid convergence to zero!
1.14 Exercises: Fourier series

1. The pulse is defined by

\[ f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}, \]

and is repeated.

(a) Calculate the coefficients of the Fourier series for this function.

(b) By substituting the value \( x = \pi/2 \) into the series give values for the errors at this point for the series truncated after;

i. one
ii. two
iii. three
iv. four
nonzero terms.

Remark. This is similar to subtracting 1/2 from Example 1 of Stewart page 3 ... and then changing its scale.

2. Using the function discussed in question 1, find an expression for the error

\[ \int_{x=\pi}^{\pi} (f(x) - g_N(x))^2 \, dx, \]

where \( g_N(x) \) is the series truncated at \( n = N \). Using the fact that

\[ \sum_{n=1}^{n=\infty} \frac{(1 - (-1)^n)^2}{n^2} = \frac{\pi^2}{2} \]

show that as \( N \to \infty \) this error tends to zero.

3. Calculate the Fourier series of the periodic function \( f(x) = x^3 \) defined between \( x = -\pi \) and \( x = \pi \).

4. Find the Fourier series of the periodic function \( f(x) = x \) defined on the interval \( x = 0 \) to \( x = 2\pi \). Calculate the error at \( x = \pi \) for the series truncated after; (a) twenty, (b) thirty and (c) forty terms. Calculate the error at \( x = \pi/2 \) for the series truncated after, (d) one, (e) two, (f) three and (g) four nonzero terms.

5. Find the Fourier series of the periodic function \( |x| \) defined on the interval \((-\pi, \pi)\).

6. Find the Fourier series of the periodic function defined by

\[ f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi/2 \\ \frac{1}{2} & \pi/2 < x < \pi \end{cases} \]
7. A function $f(x) = x$ is only defined on the interval $x = (0, \pi)$ extend the function (a) periodically, (b) as an even function and (c) as an odd function, and hence determine the corresponding Fourier transforms.

By truncating each series after four (nonzero) terms compare the errors at $x = \frac{\pi}{2}$ and $x = \pi$.

8. Repeat question 7 where $f(x) = 1 - x$, with the interval $x = (0, 1)$ finding the errors at $x = \frac{1}{2}$ and $x = 1$.

1.14.1 Solutions: Fourier series

1. (a) $2L = 2\pi$ as the function is odd all the cosine components will be zero and

$$b_n = \frac{1}{\pi} \int_{x=-\pi}^{x=\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{x=-\pi}^{0} -\sin nx \, dx + \frac{1}{\pi} \int_{x=0}^{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi n} \left(1 - (-1)^n\right),$$

so that

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \left(1 - (-1)^n\right) \sin nx.$$ 

(b)

$$n = 1 \quad \frac{4}{\pi} \approx 1.2732 \quad E = 0.27$$

$$n = 3 \quad \frac{4}{\pi} - \frac{4}{3\pi} \approx 0.8488 \quad E = 0.1512$$

$$n = 5 \quad \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} \approx 1.10347 \quad E = 0.103$$

$$n = 7 \quad \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} - \frac{4}{7\pi} \approx 0.9215 \quad E = 0.078$$

2.

$$\int_{x=-\pi}^{x=\pi} (f(x) - g(x))^2 \, dx = \int_{x=-\pi}^{0} \left(-1 - \sum_{n=1}^{N} \frac{2}{\pi n} (1 - (-1)^n) \sin nx\right)^2 \, dx$$

$$+ \int_{x=0}^{\pi} \left(1 - \sum_{n=1}^{N} \frac{2}{\pi n} (1 - (-1)^n) \sin nx\right)^2 \, dx$$

We can write

$$\int_{x=-\pi}^{0} \left(-1 - \sum_{n=1}^{N} \frac{2}{\pi n} (1 - (-1)^n) \sin nx\right)^2 \, dx = \int_{x=\pi}^{0} \left(-1 + \sum_{n=1}^{N} \frac{2}{\pi n} (1 - (-1)^n) \sin nz\right)^2 \, (-dz)$$

$$= \int_{x=0}^{\pi} \left(1 - \sum_{n=1}^{N} \frac{2}{\pi n} (1 - (-1)^n) \sin nz\right)^2 \, dz.$$
1.14. EXERCISES: FOURIER SERIES

Hence, using the short-hand $b_n$ for the coefficient,

$$E = 2 \left( \int_0^\pi 1 - 2 \sum_{n=1}^N b_n \sin nx + \left( \sum_{n=1}^N b_n \sin nx \right)^2 \, dx \right).$$

Since the integral of $\sin nx \sin mx$ is zero over this interval, except when $n = m$, we have

$$\int_0^\pi \left( \sum_{n=1}^N b_n \sin nx \right)^2 \, dx = \int_0^\pi \sum_{n=1}^N b_n^2 \sin^2 nx \, dx.$$

It then follows that

$$E = 2\pi - \pi \sum_{n=1}^N b_n^2$$

but we are given that

$$\sum_{n=1}^\infty \frac{(1 - (-1)^n)^2}{n^2} = \frac{\pi^2}{2}$$

so as $N \to \infty$ it is easy to show that $E = 0$.

3. The function is odd so that all the cosine coefficients are zero including $a_0$.

$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi x^3 \sin nx \, dx = (-1)^n \left( \frac{12}{n^3} - \frac{2\pi^2}{n} \right),$$

so that

$$f(x) = \sum_{n=1}^\infty (-1)^n \left( \frac{12}{n^3} - \frac{2\pi^2}{n} \right) \sin nx.$$

4. $f(x) = x \,(0, 2\pi)$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x \, dx = \pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx \, dx = -\frac{2}{n}$$

so that

$$f(x) = \pi - 2 \sum_{n=1}^\infty \frac{2}{n} \sin nx.$$

error at $x = \pi$ is zero independent of number of terms taken since the only nonzero term is $a_0$. Error at $x = \pi/2$

$$n = 0 \quad \pi - \frac{\pi}{2} \approx 1.57$$

after one more nonzero term \quad $\pi - 2 - \frac{\pi}{2} \approx -0.429$
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after one more nonzero term \( \pi - 2 + \frac{2}{3} - \frac{\pi}{2} \approx 0.23746 \)

after one more nonzero term \( \pi - 2 + \frac{2}{3} - \frac{2}{5} - \frac{\pi}{2} \approx -0.16253 \)

5. The function is even so that all the sine coefficients will be zero.

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{1}{2\pi} \left( \int_{-\pi}^{0} -x \, dx + \int_{0}^{\pi} x \, dx \right) = \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \frac{\pi}{2}
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx \, dx = \frac{2}{\pi n^2}((-1)^n - 1)
\]

so

\[f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}((-1)^n - 1) \cos nx.\]

6.

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{0}^{\pi} x \, dx + \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{x}{2} \, dx = \frac{3}{8}
\]

\[
a_n = \frac{1}{\pi} \int_{0}^{\pi} \cos nx \, dx + \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} \cos nx \, dx = \frac{1}{2\pi n} \sin \frac{n\pi}{2}
\]

\[
b_n = \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx + \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} \sin nx \, dx = \frac{1}{\pi n} \left(1 - \frac{(-1)^n}{2} - \frac{1}{2} \cos \frac{n\pi}{2}\right)
\]

so that

\[f(x) = \frac{3}{8} + \sum_{n=1}^{\infty} \frac{1}{2\pi n} \sin \frac{n\pi}{2} \cos nx + \frac{1}{\pi n} \left(1 - \frac{(-1)^n}{2} - \frac{1}{2} \cos \frac{n\pi}{2}\right) \sin nx\]

7. \(f(x) = x\) \((0, \pi)\) so that \(L = \pi/2\)

\[
a_0 = \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \frac{\pi}{2}
\]

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} x \cos 2nx \, dx = 0
\]

\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin 2nx \, dx = -\frac{1}{n}
\]

\[f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{1}{n} \sin 2nx\]
Odd extension: $f(x) = x - \pi < x < \pi$, obviously all the cosine coefficients are going to be zero. $L = \pi$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = -\frac{2(-1)^n}{n}$$

$$f_o(x) = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n} \sin nx$$

Even extension: $f(x) = -x -\pi < x < 0$ and $f(x) = x$ when $0 < x < \pi$ again $L = \pi$

$$a_0 = \frac{1}{\pi} \int_{0}^{\pi} x \, dx = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx \, dx = \frac{4}{\pi n^2} ((-1)^n - 1)$$

$$f_e(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} ((-1)^n - 1) \cos nx$$

Errors at $x = \pi/2$:

Full solution: Exact, Odd extension: $124/105 - \frac{\pi}{2}$, and the even extension is exact. and at $x = \pi$ Full solution: error is $\pi/2$, odd extension error is $\pi$ and even extension error is $1036/(225\pi) - \pi/2$.

Overall the even solution seems to win out.

8. $f(x) = 1 - x$, $L = 1/2$

$$a_0 = \int_{0}^{1} 1 - x \, dx = \frac{1}{2}$$

$$a_n = 2 \int_{0}^{1} (1 - x) \cos 2n\pi x \, dx = 0$$

$$b_n = 2 \int_{0}^{1} (1 - x) \sin 2n\pi x \, dx = \frac{1}{\pi n}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin 2n\pi x$$

Even extension: $f(x) = 1 - x$ $0 < x < 1$ and $f(x) = 1 + x$ for $-1 < x < 0$, all the sine coefficients are zero.

$$a_0 = \frac{2}{2} \int_{0}^{1} f(x) \, dx = \frac{1}{2}$$

$$a_n = 2 \int_{0}^{1} (1 - x) \cos n\pi x \, dx = -\frac{2}{(\pi n)^2} ((-1)^n - 1)$$

$$f_e(x) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{2}{(\pi n)^2} ((-1)^n - 1) \cos n\pi x$$
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Figure 1.6: RLC-circuit.

and for the odd extension \( f(x) = -(1 - x) \) for \(-1 < x < 0\), clearly all the cosine coefficients are zero.

\[
b_n = 2 \int_0^1 (1 - x) \sin n\pi x \, dx = \frac{2}{n\pi}
\]

so that

\[
f_0(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi x
\]

Errors at \( x = \frac{1}{2} \): Full solution is exact, odd extension \( \frac{124}{(105\pi)} - \frac{1}{2} \), and the even extension is exact. For \( x = 1 \) full solution is always ‘wrong’ by \( \frac{1}{2} \), odd extension is exact, and the even extension is ‘wrong’ by \( \frac{1}{2} - \frac{1036}{(225\pi^2)} \). (Translate ‘wrong’ here as ‘departs from the formula defining the function on the little bit of domain, before its extension to the whole line’.)

1.15 O.D.E. Applications: the RLC-circuit

The RLC-circuit in Fig 1.6 is governed by the second order differential equation

\[
L \frac{d^2}{dt^2} i(t) + R \frac{di(t)}{dt} + \frac{1}{C} i = \frac{d}{dt} v(t).
\]

(1.12)

We will look at the case with \( L = 1 \) Henry, \( R = 0.02 \) Ω and \( C = 0.04 \) Farad. The electromotive force \( v(t) \) is given by

\[
v(t) = \begin{cases} 
\frac{1}{2} t^2 + \frac{\pi}{2} t & \text{if } -\pi \leq t \leq 0, \\
-\frac{1}{2} t^2 + \frac{\pi}{2} t & \text{if } 0 \leq t \leq \pi,
\end{cases}
\]

with \( v(t) \) periodic with a Period \( P = 2\pi \).

First, we write the right hand side of (1.12) as a Fourier series. The derivative of \( v(t) \) takes the form

\[
\frac{d}{dt} v(t) = \begin{cases} 
t + \frac{\pi}{2} & \text{if } -\pi \leq t \leq 0, \\
-t + \frac{\pi}{2} & \text{if } 0 \leq t \leq \pi,
\end{cases}
\]
and can be represented by the Fourier series (the derivative of \( v(t) \) is an even function!)

\[
\frac{d}{dt} v(t) = \frac{4}{\pi} \left( \cos(t) + \frac{1}{3^2} \cos(3t) + \frac{1}{5^2} \cos(5t) + \ldots \right),
\]

(1.13)

with as general term

\[
\frac{4}{n^2 \pi} \cos(nt) \quad \text{for } n = 1, 3, 5, \ldots
\]

Then, we write the solution in terms of a Fourier series as

\[
i(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)),
\]

(1.14)

For periodic functions that are continuous everywhere, we can approximate the derivative by differentiating the Fourier series term by term,

\[
\frac{d}{dt} i(t) = \sum_{n=1}^{\infty} \left( -a_n n \sin(nt) + b_n n \cos(nt) \right).
\]

(1.15)

Similarly, (assuming the derivative of \( i(t) \) is a continuous function) we can write

\[
\frac{d^2}{dt^2} i(t) = \sum_{n=1}^{\infty} \left( -a_n n^2 \cos(nt) - b_n n^2 \sin(nt) \right),
\]

(1.16)

If we substitute the expressions (1.13), (1.14) – (1.16) into the differential equation (1.12), and combine identical cosine and sine terms on the left hand side, we obtain the equation

\[
\sum_{n=1}^{\infty} \left( \left( -Ln^2 a_n + Rnb_n + \frac{1}{C}a_n \right) \cos(nt) \\
+ \left( -Ln^2 b_n - Rna_n + \frac{1}{C}b_n \right) \sin(nt) \right) = \frac{4}{\pi} \left( \cos(t) + \frac{1}{3^2} \cos(3t) + \frac{1}{5^2} \cos(5t) + \ldots \right).
\]

Comparing the terms for even \( n \)-values, we find that

\[
\left( -Ln^2 + \frac{1}{C} \right) a_n + nRb_n = 0,
\]

\[
-Rna_n + \left( -Ln^2 + \frac{1}{C} \right) b_n = 0,
\]

which is only solved by \( a_n = b_n = 0 \). For odd values of \( n \), we find

\[
\left( -Ln^2 + \frac{1}{C} \right) a_n + nRb_n = \frac{4}{n^2 \pi},
\]

\[
-Rna_n + \left( -Ln^2 + \frac{1}{C} \right) b_n = 0,
\]

which, for the values given above, has the solution

\[
a_n = \frac{4(25 - n^2)}{n^2 \pi D_n},
\]

\[
b_n = \frac{0.08}{n \pi D_n},
\]
Figure 1.7: The solution $i(t)$ as approximated by the first six frequency components of its Fourier series.

with

$$D_n = (25 - n^2)^2 + (0.02n)^2.$$  

We can study each frequency component in the resulting Fourier series rather than plotting the overall solution. In particular, it is useful to look at the amplitude of each of the frequency components, i.e., the combinations

$$a_n \cos(nt) + b_n \sin(nt). \quad (1.17)$$

The amplitude of a frequency component of the form (1.17) is given by the relation

$$c_n = \sqrt{a_n^2 + b_n^2},$$

so that for the given circuit, and for $n$ odd,

$$c_n = \frac{4}{n^2 \pi \sqrt{D_n}}.$$  

If we calculate these amplitudes, one finds

<table>
<thead>
<tr>
<th></th>
<th>$c_1$</th>
<th>$c_3$</th>
<th>$c_5$</th>
<th>$c_7$</th>
<th>$c_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0530</td>
<td>0.0088</td>
<td>0.5100</td>
<td>0.0011</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

It is clear from these values that the term with $n = 5$ is dominating the solution. Fig 1.7 depicts the Fourier series of $i(t)$ truncated after the sixth frequency component, i.e., the terms with $\sin(11t)$ and $\cos(11t)$. The dominance of the third frequency component ($n = 5$) is obvious.
1.16 Exercises: Fourier series and O.D.E.s

1. Find the Fourier series of the current \(I(t)\) in the RLC–circuit with \(R = 100\) Ohms, \(L = 10\) henrys, \(C = 0.01\),

\[
v(t) = \begin{cases} 
100(\pi t + t^2) & -\pi \leq t \leq 0 \\
100(\pi t - t^2) & 0 \leq t \leq \pi 
\end{cases}
\]

where \(v(t)\) is a periodic function with a period \(p = 2\pi\).

2. Show that the periodic function

\[f(t) = t \quad -T < t < T \quad \text{with} \quad f(t + 2T) = f(t)\]

has the Fourier series expansion

\[
f(t) = \frac{2T}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi t}{T}
\]

By integrating the series term–by–term show that the periodic function

\[g(t) = t^2 \quad -T < t < T \quad \text{with} \quad g(t + 2T) = g(t)\]

has a Fourier expansion

\[
g(t) = \frac{T^2}{3} - \frac{4T^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi t}{T}
\]

1.16.1 Solutions: Fourier series and O.D.E.s

1. So that

\[
\frac{dv}{dt} = \begin{cases} 
100(\pi + 2t) & -\pi \leq t \leq 0 \\
100(\pi - 2t) & 0 \leq t \leq \pi 
\end{cases}
\]

since the function we will only have the cosine coefficients. \(\tilde{a}_0 = 0\)

\[
\tilde{a}_n = \frac{200}{\pi} \int_{-\pi}^{\pi} (\pi - 2t) \cos nt dt = \frac{400}{\pi n^2} (1 - (-1)^n)
\]

Recall that the current satisfies

\[
L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dv}{dt}
\]

try a solution of the form

\[
i = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt,
\]

substituting into the differential equation and equating coefficients of cosines and sines (at each \(n\)) (initially we find \(a_0 = 0\))

\[-n^2 a_n L + nb_n R + \frac{1}{C} a_n = \frac{400}{\pi n^2} (1 - (-1)^n)\]
\[-n^2 b_n L - na_n R + \frac{1}{C} b_n = 0\]

solving these we have

\[b_n = \frac{\frac{400R}{\pi n} (1 - (-1)^n)}{\left(\frac{-n^2 L + \frac{1}{\beta}}{2} + n^2 R^2\right)}\]

\[a_n = \frac{\frac{400}{\pi n^2} \left(-n^2 L + \frac{1}{\beta}\right) (1 - (-1)^n)}{\left(\frac{-n^2 L + \frac{1}{\beta}}{2} + n^2 R^2\right)}\]

we can then substitute the values from the question.

2. The function is odd so we only need the sine coefficients

\[b_n = \frac{2}{T} \int_0^T t \sin \frac{n\pi t}{T} dt = -\frac{2T}{n\pi} (-1)^n\]

integrate

\[\int f(t) dt = \frac{g(t)}{2} = \frac{C}{2} + \frac{2T}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{T}{n\pi} \cos \frac{nt\pi}{T}\]

the value of C is given by \(a_0\)

\[a_0 = \frac{2}{2T} \int_0^T t^2 dt = \frac{T^2}{3}\]

1.17 Introduction: Convergence issues

These notes merely alert you to some of the issues. No convergence matters will be engineering-maths exams.

In earlier units you have treated the convergence of sequences and series of real numbers. There are AiM Revision Quizzes on these topics (worth 0 marks) for those of you who want to do these.

**Definition.** We say that a sequence \((\xi_m)\) converges to limit \(L\) if

\[\forall \epsilon > 0 \; \exists M_\epsilon \; \text{such that} \; m > M_\epsilon \implies |\xi_m - L| < \epsilon .\]

We also write \(\xi_m \to L\) as \(m \to \infty\).

For infinite series we say that the series \(\sum_{k=0}^{\infty} \alpha_k\) is convergent (to \(L\)) if the partial sums \(\sigma_m = \sum_{k=0}^{m} \alpha_k\) converge (to \(L\)), and for such a convergent series we write

\[\sum_{k=0}^{\infty} \alpha_k = L\ .\]

In earlier units you also treated power series, which can be viewed as an example of a series of functions. The general setting for series of functions is to be given a sequence of functions \(\phi_n(x)\) and to be concerned with series of the form \(\sum_{n=0}^{\infty} \alpha_n \phi_n(x)\). In the power series work you had
$\phi_n(x) = x^n$: in the present Fourier series topic the $\phi_n$ are trig. functions (sine and/or cosine). In your earlier units, amongst other examples of power series, you saw

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } -1 < x < 1$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x) \quad \text{for all } x .$$

You also learnt that termwise differentiation of a power series ‘works’ in the desirable way, for $x$ in the interval of convergence. The issues with Fourier series are a bit different: we are not free to choose interval for the convergence.

1.18 Convergence in inner-product spaces

1.18.1 Convergence in the Euclidean space $\mathbb{R}^n$

$\mathbb{R}^n$ with its usual dot-product is an inner-product space. The length of $x \in \mathbb{R}^n$ is defined by

$$\| x \| = \sqrt{x \cdot x} .$$

Just as in your earlier work on convergence of sequences of numbers, one can define convergence of sequences of vectors.

**Definition.** Let $V$ be an inner-product space. We say that a sequence $(\xi_m)$, with $\xi_m \in V$, **converges to limit** $L$ if

$$\forall \epsilon > 0 \exists M_\epsilon \text{ such that } m > M_\epsilon \implies |\xi_m - L| < \epsilon .$$

**EXAMPLE** $V = \mathbb{R}^2$. $\xi_m = (\frac{1}{m}, \frac{1}{2\pi})$. We have (pretty obviously) $\xi_m \to 0$.

1.18.2 Convergence in any inner-product space

Though our example above was just ‘points in the plane’, the definition was written to work in any inner-product space, whether finite dimensional or not. (It is conventional to bold-face vectors in $\mathbb{R}^n$ but not in general.)

1.19 Completeness of inner-product spaces

This section is very definitely not in engineering-maths exams! However, you should know that Fourier, early in the nineteenth century, had a difficult job convincing the world that Fourier series work.

**Definition.** Let $V$ be an inner-product space and $(\xi_m)$ a sequence in $V$. The sequence $(\xi_m)$ is said to be **Cauchy** if

$$\forall \epsilon > 0 \exists M_\epsilon \text{ such that } n > m > M_\epsilon \implies |\xi_n - \xi_m| < \epsilon .$$

Let $V$ be an inner-product space. If every Cauchy sequence of elements of $V$ is Cauchy, we say that $V$ is **complete**.

**EXAMPLES.**

$\mathbb{R}^n$ is complete (using the usual inner product).

$\mathbb{Q}^n$ (the $n$-tuples of rational numbers) is not complete. Consider now the inner-product for $2\pi$-
periodic functions:
\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx . \]

The space of continuous $2\pi$-periodic functions is not complete.
$L_2$ is complete.
We have in earlier chapters seen examples of Fourier Series for piecewise continuous functions with jump discontinuities.

1.19.1 The main facts for Fourier’s theory

FACT 1. Let $V$ be the inner-product space of square-integrable functions defined on $(a, b)$ with
\[ \langle f, g \rangle = \int_{a}^{b} f(x)g(x) \, dx . \]

Then $V$ is a complete inner-product space.

FACT 2. Let $V$ be the (complete) inner-product space of square-integrable functions defined on $(-\pi, \pi)$ with
\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx . \]

Let
\[ \mathcal{B} = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots \} . \]

Then $\mathcal{B}$ is an orthogonal basis for $V$.

Another way to say Fact 2 is that the Fourier series of any square integrable function converges to the function in the (complete) inner-product space of (all) square-integrable functions.

The speed at which the Fourier coefficients $a_n, b_n$ go to zero is an issue in this. Consider sequences $(\alpha_n), (\beta_n)$ and define
\[ \langle \alpha, \beta \rangle_{\text{seq}} = \sum_{n} \alpha_n \beta_n . \]

The set of all sequences $\alpha$ such that $\langle \alpha, \beta \rangle_{\text{seq}}$ is bounded is denoted $\ell_2$. $\ell_2$ is a complete inner product space.

FACT. The Fourier coefficients of $L_2$ functions are in $\ell_2$, i.e.
\[ a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty . \]

1.20 Advanced topics: pointwise convergence, etc.

The sine and cosine functions appearing in the Fourier series are continuous. Hence all the partial sums are continuous. However, we have already seen examples where the limit of the Fourier series is not continuous. See the Stewart Supplement where Fourier’s Pointwise Convergence Theorem, box 8, page 5, is stated without proof. (Incidentally, it wasn’t proved until after Fourier’s time.)

In applications it frequently happens that we want to perform calculus operations term-by-term. Differentiation, in particular, makes the terms in the series tend to zero more slowly, so reduces the likelihood of the series converging.
There are results in the area (and these are in the large AEM, *Advanced Engineering Maths*, books, e.g. O’Neil, Chapter 14). These are best derived from building from the convergence in the mean-square results we discussed in the earlier sections of this chapter, but they are, compared to the $L_2$-convergence results, more difficult to derive.

It is our job in engineering maths units to alert you to always making the obvious checks. If you have a series of sine and cosine terms looking like it might be a Fourier series, first check that its terms $a_n$, $b_n$ tend to zero as $n$ tends to infinity. If they don’t, you have a divergent series and you have to think again about how to try to do your calculation.
Chapter 2

Fourier Integrals

The Fourier series can be applied to periodic functions or periodically expanded functions. What can we do if we have a nonperiodic function? We will examine that by looking at what happens to the Fourier series when we let the period of a periodic function become infinitely large.

2.1 Square waves with increasing periods

Consider the periodic square wave

\[ f_L(t) = \begin{cases} 
0 & \text{if } -L \leq t < -1, \\
1 & \text{if } -1 \leq t \leq 1, \\
0 & \text{if } 1 < t \leq L.
\end{cases} \]

The period of this square wave is \( p = 2L \).

In Fig. 2.1 and Fig. 2.2 the above defined square wave is depicted for a period \( L = 4 \) and \( L = 12 \) respectively. If we let the period become very large, only a few pulses of the square wave will occur in a finite amount of time. If we take the limit of an infinite period, we obtain a single pulse:

\[ f(t) = \lim_{L \to \infty} f_L(t) = \begin{cases} 
1 & \text{if } -1 \leq t \leq 1, \\
0 & \text{otherwise}.
\end{cases} \]

Figure 2.1: The periodic square wave for a period of 4 units.
2.2 The Fourier series for increasing period

Since the periodic square wave is even, we can use the Fourier cosine series for which the coefficients are given by

\[
a_0 = \frac{1}{2L} \int_{-1}^{1} dt = \frac{1}{L}, \\
a_n = \frac{1}{L} \int_{-1}^{1} \cos \left( \frac{n\pi t}{L} \right) dt \\
\quad = \frac{2}{L} \int_{0}^{1} \cos \left( \frac{n\pi t}{L} \right) dt \\
\quad = \frac{2}{L} \sin \left( \frac{n\pi}{L} \right).
\]

The Fourier series is a sum of products of the form \( a_n \cos(n\pi t/L) \) for \( n = 0, \ldots, \infty \). We can estimate how many terms are needed by looking at the amplitude spectrum \( a_n(\omega_n) \), where \( \omega_n = n\pi/L \).

For \( L = 4 \), the different values of \( \omega_n \) are well spaced out and only a few are needed to obtain a good approximation. This is illustrated by plotting the amplitude spectrum (See Fig. 2.3). For \( L = 12 \), the different values for \( \omega_n \) are closer together and more different values are needed to get a good approximation. This is illustrated in Fig. 2.4.

For \( L \to \infty \), the values of \( \omega_n \) will be densely packed and one will need a sum over a very large number to obtain a reasonable approximation.

Using this notation, one can write the Fourier series as a sum over the different values of \( \omega_n \):

\[
f_L(t) = \frac{1}{L} \sum_{\omega_n=0}^{\infty} \tilde{a}(\omega_n) \cos(\omega_n t), \quad (2.1)
\]
2.3. THE FOURIER INTEGRAL

Figure 2.4: The amplitude spectrum for a period of 12 units.

with \( \tilde{a}(\omega_n) = La(\omega_n) = La_n \). Since

\[
\delta\omega_n = \omega_n - \omega_{n-1} = \frac{n\pi}{L} - \frac{(n-1)\pi}{L} = \frac{\pi}{L}
\]

is not dependent on \( n \), we can write the Fourier series (2.1) as

\[
f_L(t) = \frac{1}{\pi} \frac{1}{\delta\omega_n} \sum_{\omega_n=0}^{\infty} \tilde{a}(\omega_n) \cos(\omega_n t) \delta\omega_n,
\]

\[
= \frac{1}{\pi} \sum_{0}^{\infty} \tilde{a}(\omega_n) \cos(\omega_n t) \delta\omega_n. \tag{2.2}
\]

The Fourier series (2.2) now represents a sum of rectangular areas under the curve defined by \( \tilde{a}(\omega_n) \cos(\omega_n t) \) seen as a function of \( \omega_n \). If we let \( L \to \infty \) then this sum will approximate an integral, and, bearing in mind that \( f(t) \) was the limit of \( f_L(t) \) for \( L \to \infty \), we can write the limit of Equation (2.2) as

\[
f(t) = \frac{1}{\pi} \int_{0}^{\infty} \tilde{a}(\omega) \cos(\omega t) d\omega.
\]

This is called the Fourier cosine integral

\( \tilde{a}(\omega_n) \) is given by

\[
\tilde{a}(\omega_n) = \int_{-L}^{L} f_L(t) \cos(\omega_n t) dt,
\]

so that in the limit \( L \to \infty \),

\[
\tilde{a}(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt.
\]

Note that \( \tilde{a}_0 \) is found by putting \( \omega = 0 \) in the integral expression for \( \tilde{a}(\omega) \).

2.3 The Fourier integral

The example in the previous section is not a mathematical proof. It merely illustrates the idea behind the Fourier integral. We will not try to give a sound mathematical derivation of the Fourier integral, but rather present you with the resulting formulae.

A general function (neither even or odd), can be approximated by a Fourier integral as

\[
f(t) = \int_{0}^{\infty} (A(\omega) \cos(\omega t) + B(\omega) \sin(\omega t)) d\omega, \tag{2.3}
\]
CHAPTER 2. FOURIER INTEGRALS

with

\[ A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \cos(\omega \nu) d\nu, \quad (2.4) \]

\[ B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \sin(\omega \nu) d\nu. \quad (2.5) \]

We have used \( \nu \) as the integration variable to avoid confusion with the variable \( t \) which corresponds with the function \( f(t) \).

The Fourier integral can only be applied to functions which yield finite values for \( A(\omega) \) and \( B(\omega) \). This means essentially that \( f(t) \) must be piecewise continuous and that

\[ \int_{-\infty}^{\infty} |f(t)| \, dt \]

must exist.

2.4 Examples

2.4.1 Exponentially decaying signal

Given is the function

\[ f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \pi e^{-t} & \text{if } t \geq 0. \end{cases} \]

Then the coefficients (2.4) and (2.5) are given by

\[ A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \cos(\omega \nu) d\nu, \]

\[ = \frac{1}{\pi} \int_{0}^{\infty} \pi e^{-\nu} \cos(\omega \nu) d\nu, \]

\[ = \left[ \frac{e^{-\nu} \sin(\omega \nu)}{\omega} \right]_{0}^{\infty} + \frac{1}{\omega} \int_{0}^{\infty} e^{-\nu} \sin(\omega \nu) d\nu, \]

\[ = 0 + \left[ -\frac{e^{-\nu} \cos(\omega \nu)}{\omega^2} \right]_{0}^{\infty} - \frac{1}{\omega^2} \int_{0}^{\infty} e^{-\nu} \cos(\omega \nu) d\nu, \]

\[ = \frac{1}{\omega^2} - \frac{1}{\omega^2} \int_{0}^{\infty} e^{-\nu} \cos(\omega \nu) d\nu. \]

Since the integral on the right hand side is the same as the left hand side integral \((A(\omega))\), we can write that

\[ \left(1 + \frac{1}{\omega^2}\right) \int_{0}^{\infty} e^{-\nu} \cos(\omega \nu) d\nu = \frac{1}{\omega^2}, \]

or,

\[ \int_{0}^{\infty} e^{-\nu} \cos(\omega \nu) d\nu = \frac{1}{1 + \omega^2}. \]

Similarly, it can be shown that

\[ B(\omega) = \int_{0}^{\infty} e^{-\nu} \sin(\omega \nu) d\nu = \frac{\omega}{1 + \omega^2}. \]
The Fourier integral representation for \( f(t) \) is thus given as
\[
f(t) = \int_0^\infty \frac{\cos(\omega t) + \omega \sin(\omega t)}{1 + \omega^2} \, d\omega.
\]  
(2.6)

There is one obvious misfit between the Fourier integral representation and the function \( f(t) \). If one evaluates the integral (2.6) for \( t = 0 \), one finds
\[
\int_0^\infty \frac{1}{1 + \omega^2} \, d\omega = \left[ \tan^{-1}(\omega) \right]_0^\infty = \frac{\pi}{2},
\]
where \( f(0) = \pi \). Notice that the Fourier integral representation cuts the jump at \( t = 0 \) in half!

### 2.4.2 A single pulse

A single pulse can be written as
\[
f(t) = \begin{cases} 
1 & \text{if } |t| \leq 1, \\
0 & \text{if } |t| > 1.
\end{cases}
\]

The coefficients in the Fourier integral are calculated as
\[
A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \cos(\omega \nu) \, d\nu = \frac{1}{\pi} \int_{-1}^{1} \cos(\omega \nu) \, d\nu = \left[ \frac{\sin(\omega \nu)}{\omega} \right]_{-1}^{1} = 2 \sin(\omega) \frac{\pi}{\omega},
\]
\[
B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \sin(\omega \nu) \, d\nu = \frac{1}{\pi} \int_{-1}^{1} \sin(\omega \nu) \, d\nu = 0.
\]

The Fourier integral is then given by
\[
f(t) = 2 \pi \int_0^\infty \frac{\sin(\omega) \cos(\omega t)}{\omega} \, d\omega.
\]

Again, we find that at the location of the jump, i.e. \( t = 1 \), the Fourier integral yields half the value of the original function:
\[
2 \pi \int_0^\infty \frac{\sin(\omega) \cos(\omega)}{\omega} \, d\omega = 2 \pi \int_0^\infty \frac{\sin(2\omega)}{2\omega} \, d\omega = \frac{1}{2}.
\]

In general, at a point where \( f(t) \) is discontinuous, the value of the Fourier integral is given by the average of the left- and right-hand limits of \( f(t) \) at that point.

Fig. 2.5 shows the single pulse and the approximation to the Fourier cosine integral
\[
\frac{2}{\pi} \int_0^{100} \frac{\sin(\omega) \cos(\omega t)}{\omega} \, d\omega.
\]  
(2.7)
2.5 Fourier cosine and sine integrals

2.5.1 Definition

The last example was an illustration of how the Fourier integral looks for an even function, i.e. it contains only terms in $\cos(\omega t)$. One has a similar result for odd functions, where only terms in $\sin(\omega t)$ occur. Therefore one speaks about a Fourier cosine integral and a Fourier sine integral.

Even functions are approximated by a Fourier cosine integral of the form

$$f(t) = \int_0^\infty A(\omega) \cos(\omega t) d\omega,$$

with

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(\nu) \cos(\omega \nu) d\nu.$$

Odd functions can be approximated by a Fourier sine integral

$$f(t) = \int_0^\infty A(\omega) \sin(\omega t) d\omega,$$

with

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(\nu) \sin(\omega \nu) d\nu.$$

2.5.2 Example

Take the function $f(t) = e^{-kt}$ for $t > 0$ and $k > 0$. We can expand this function into an even function and apply the Fourier cosine integral. The coefficient $A(\omega)$ is found as

$$A(\omega) = \frac{2}{\pi} \int_0^\infty e^{-k\nu} \cos(\omega \nu) d\nu = \frac{2k}{\pi k^2 + \omega^2}.$$
The Fourier cosine integral can then be written as

\[ f(t) = \frac{2k}{\pi} \int_0^\infty \frac{\cos(\omega t)}{k^2 + \omega^2} d\omega. \]

for \( t > 0 \) and \( k > 0 \). In this way we have determined the result of the Laplace integral

\[ \int_0^\infty \frac{\cos(\omega t)}{k^2 + \omega^2} d\omega = \pi \frac{e^{-kt}}{2k}, \]

for \( t > 0 \) and \( k > 0 \).

If one expands \( f(t) \) into an odd function and applies the Fourier sine integral, one finds the results for the second Laplace integral:

\[ \int_0^\infty \frac{\omega \sin(\omega t)}{k^2 + \omega^2} d\omega = \frac{\pi}{2} e^{-kt}, \]

for \( t > 0 \) and \( k > 0 \).
Chapter 3

Fourier Cosine and Sine Transforms

We have seen that an even (odd) function can be approximated by a Fourier cosine (sine) integral. This integral representation can be used to define the *Fourier cosine (sine) transform*. In general, an integral transform is a transformation of a given function into another function which depends on another variable. They are used because they help us to solve ordinary and partial differential equations.

The Matlab Symbolic Toolbox has commands *fourier* and *ifourier* from which it is possible – with a bit of work involving (i) even or odd extensions, (ii) taking real and imaginary parts – to derive Fourier cosine and Fourier sine transforms.

3.1 The Fourier cosine transform

The even function $f(t)$ can be approximated by the Fourier cosine integral,

$$f(t) = \int_0^\infty A(\omega) \cos(\omega t) d\omega,$$

with

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(\nu) \cos(\omega \nu) d\nu.$$

Let us define the function $\hat{f}_c(\omega)$ as

$$\hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}} A(\omega).$$

Replacing the integration variable $\nu$ by $t$, we can then write

$$\hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}} \int_0^\infty f(t) \cos(\omega t) dt,$$

$$f(t) = \sqrt{\frac{\pi}{2}} \int_0^\infty \hat{f}_c(\omega) \cos(\omega t) d\omega.$$

In Equation (3.1) we define the *Fourier cosine transform* of the function $f(t)$. Equation (3.2) defines the *inverse Fourier cosine transform* of $\hat{f}_c(\omega)$. The index $c$ stands for ‘cosine’. One also uses the notation $\mathcal{F}_c(f)$ instead of $\hat{f}_c(\omega)$. 
CHAPTER 3. FOURIER COSINE AND SINE TRANSFORMS

3.2 The Fourier sine transform

We can approximate the odd function \( f(t) \) by the Fourier sine integral,

\[
f(t) = \int_0^\infty B(\omega) \sin(\omega t) d\omega,
\]

with

\[
B(\omega) = \frac{2}{\pi} \int_0^\infty f(\nu) \sin(\omega \nu) d\nu.
\]

Now, define the function \( \hat{f}_s(\omega) \) as

\[
\hat{f}_s(\omega) = \sqrt{\frac{\pi}{2}} B(\omega).
\]

Replacing \( \nu \) by \( t \), we find,

\[
\hat{f}_s(\omega) = \sqrt{\frac{\pi}{2}} \int_0^\infty f(t) \sin(\omega t) dt, \quad (3.3)
\]

\[
f(t) = \sqrt{\frac{\pi}{2}} \int_0^\infty \hat{f}_s(\omega) \sin(\omega t) d\omega. \quad (3.4)
\]

Equation (3.3) defines the Fourier sine transform of the function \( f(t) \), while Equation (3.4) defines the inverse Fourier sine transform of \( \hat{f}_s(\omega) \). The index \( s \) stands for 'sine'. An alternative notation for the Fourier sine transform is \( \mathcal{F}_s(f) \).

3.3 Examples

3.3.1 Fourier cosine transform

Take the function

\[
f(t) = \begin{cases} k & \text{if } 0 < t \leq a, \\ 0 & \text{if } a < t. \end{cases}
\]

If we consider an even extension for \( t < 0 \), we can calculate the Fourier cosine transform as

\[
\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}} \int_0^\infty f(t) \cos(\omega t) dt,
\]

\[
= \sqrt{\frac{\pi}{2}} k \int_0^a \cos(\omega t) dt,
\]

\[
= \frac{\sqrt{2} k}{\omega} \sin(a\omega),
\]
3.3. EXAMPLES

The Fourier cosine transform is a smooth function of $\omega$ as can be seen from Fig. 3.1.

The Fourier cosine transform of the (even extension of the) function $g(t) = e^{-t}$ for $0 \leq t$ is found to be

$$ \mathcal{F}_c(g) = \hat{g}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \cos(\omega t) \, dt, $$

$$ = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \cos(\omega t) \, dt, $$

$$ = \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-t} \sin(\omega t)}{\omega} \right]_0^\infty + \frac{1}{\omega} \int_0^\infty e^{-t} \sin(\omega t) \, dt, $$

$$ = \sqrt{\frac{2}{\pi}} \left[ 0 + \left[ \frac{e^{-t} \cos(\omega t)}{\omega^2} \right]_0^\infty - \frac{1}{\omega^2} \int_0^\infty e^{-t} \cos(\omega t) \, dt \right]. $$

Therefore, the Fourier cosine transform of $g(t)$ is given by

$$ \mathcal{F}_c(g) = \hat{g}_c(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2}. $$

Again, this is a smooth function of $\omega$. It is depicted in Fig. 3.2.

3.3.2 Fourier sine transform

Now, we choose an odd extension ($t < 0$) for the function $f(t) = \begin{cases} k & \text{if } 0 < t \leq a, \\ 0 & \text{if } a < t. \end{cases}$

The Fourier sine transform is given by

$$ \mathcal{F}_s(f) = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(\omega t) \, dt, $$

$$ = \sqrt{\frac{2}{\pi}} k \int_0^a \sin(\omega t) \, dt, $$

$$ = \sqrt{\frac{2}{\pi}} k \frac{1 - \cos(\omega a)}{\omega}. $$

The Fourier sine transform of $f(t)$ is depicted in Fig 3.3. The Fourier sine transform of the (odd
CHAPTER 3. FOURIER COSINE AND SINE TRANSFORMS

Figure 3.3: The Fourier sine transform of the function $f(t)$ for $k = 3$ and $a = 5$.

Figure 3.4: The Fourier sine transform of the function $g(t)$.

extension of the) function $g(t) = e^{-t}$ for $0 \leq t$ can be found in a similar way as the Fourier cosine transform for $g(t)$:

$$
\mathcal{F}_s(g) = \hat{g}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \sin(\omega t) dt,
= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \sin(\omega t) dt,
= \sqrt{\frac{2}{\pi}} \frac{\omega}{1 + \omega^2}.
$$

The Fourier sine series for the exponentially decaying signal $g(t)$ is depicted in Fig 3.4.

3.4 Linearity

If we have two functions $f(t)$ and $g(t)$ for which the Fourier cosine transform exists, than we can write the Fourier cosine transform of the linear combination $af(t) + bg(t)$ as

$$
\mathcal{F}_c(af + bg) = \sqrt{\frac{2}{\pi}} \int_0^\infty [af(t) + bg(t)] \cos(\omega t) dt,
= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(\omega t) dt + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \cos(\omega t) dt,
= a\mathcal{F}_c(f) + b\mathcal{F}_c(g).
$$

In a similar way, one can show that

$$
\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g).
$$
3.5 The transforms of a derivative

If \( f(t) \) is a function which has a Fourier cosine (sine) transform and \( f(t) \to 0 \) for \( t \to \infty \), then the Fourier cosine (sine) transform of the derivative is given by

\[
\mathcal{F}_c\left(\frac{d}{dt}f\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{dt}f(t) \cos(\omega t) dt,
\]

\[
= \sqrt{\frac{2}{\pi}} \left( [f(t) \cos(\omega t)]_0^\infty + \omega \int_0^\infty f(t) \sin(\omega t) dt \right),
\]

\[
= -\sqrt{\frac{2}{\pi}} f(0) + \omega \mathcal{F}_s(f).
\]

Similarly

\[
\mathcal{F}_s\left(\frac{d}{dt}f\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{dt}f(t) \sin(\omega t) dt,
\]

\[
= \sqrt{\frac{2}{\pi}} \left( [f(t) \sin(\omega t)]_0^\infty - \omega \int_0^\infty f(t) \cos(\omega t) dt \right),
\]

\[
= -\omega \mathcal{F}_c(f).
\]

We can summarise these results as

\[
\mathcal{F}_c(f') = \omega \mathcal{F}_s(f) - \sqrt{\frac{2}{\pi}} f(0),
\]

\[
\mathcal{F}_s(f') = -\omega \mathcal{F}_c(f),
\]

where we have written the derivative as \( f'(t) \).

We can apply these formulae to the second derivative assuming that the derivative tends to zero for \( t \to \infty \).

\[
\mathcal{F}_c\left(\frac{d^2}{dt^2}f\right) = \omega \mathcal{F}_s(f') - \sqrt{\frac{2}{\pi}} f'(0)
\]

\[
= -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0).
\]

Similarly

\[
\mathcal{F}_s\left(\frac{d^2}{dt^2}f\right) = -\omega \mathcal{F}_c(f')
\]

\[
= -\omega^2 \mathcal{F}_s(f) + \omega \sqrt{\frac{2}{\pi}} f(0).
\]

We can use these properties to facilitate the calculation of Fourier cosine (sine) transforms. To calculate the Fourier cosine transform of (the even extension of) \( e^{-at} \) with \( a > 0 \), we use the property that

\[
\frac{d^2}{dt^2} \left( e^{-at} \right) = a^2 e^{-at}.
\]

Using the property (3.5) we find that

\[
a^2 \mathcal{F}_c\left( e^{-at} \right) = -\omega^2 \mathcal{F}_c\left( e^{-at} \right) - \sqrt{\frac{2}{\pi}} \left( -ae^{-at} \right),
\]
or

\[
(a^2 + \omega^2) \mathcal{F}_c \left( e^{-at} \right) = \sqrt{\frac{2}{\pi}} a,
\]

so that

\[
\mathcal{F}_c \left( e^{-at} \right) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}.
\]
Chapter 4

The Fourier Transform

The Fourier cosine and sine transforms are special cases of the Fourier transform which will prove extremely useful in solving partial differential equations.

The first section of this chapter provides a link with the previous chapter. In it, we need to rewrite the Fourier integral in a complex form (similar to the complex Fourier series).

4.1 Complex Fourier integral

The Fourier integral

\[ f(t) = \int_{0}^{\infty} [A(\omega) \cos(\omega t) + B(\omega) \sin(\omega t)] d\omega, \]

with

\[ A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \cos(\omega \nu) d\nu, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\nu) \sin(\omega \nu) d\nu. \]

When we substitute the equalities

\[ \cos(\omega t) = \frac{1}{2} \left( e^{i\omega t} + e^{-i\omega t} \right), \quad \sin(\omega t) = \frac{1}{2i} \left( e^{i\omega t} - e^{-i\omega t} \right), \]

into the expression for the Fourier integral, we obtain the complex Fourier integral:

\[ f(t) = \int_{0}^{\infty} \left( C(\omega) e^{i\omega t} + D(\omega) e^{-i\omega t} \right) d\omega \]

with

\[ C(\omega) = \frac{1}{2} \left( A(\omega) - iB(\omega) \right) \]
\[ = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} f(\nu) \cos(\omega \nu) d\nu - i \int_{-\infty}^{\infty} f(\nu) \sin(\omega \nu) d\nu \right) \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\nu) e^{-i\omega \nu} d\nu, \]

\[ D(\omega) = \frac{1}{2} \left( A(\omega) + iB(\omega) \right) \]
\[ = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} f(\nu) \cos(\omega \nu) d\nu + i \int_{-\infty}^{\infty} f(\nu) \sin(\omega \nu) d\nu \right) \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\nu) e^{i\omega \nu} d\nu, \]
\[ = C(-\omega) \]
Figure 4.1: The real part of the Fourier transform of the square pulse with $k = 2$ and $a = 4$.

So we can write the Fourier integral in complex form as

$$f(t) = \int_{-\infty}^{\infty} C(\omega)e^{i\omega t} d\omega,$$

with

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\nu)e^{-i\omega \nu} d\nu.$$

Most functions can be approximated by their Fourier integral. This forms the basis for the definition of the Fourier transform.

### 4.2 The Fourier transform

We define the function $\hat{f}(\omega)$ as

$$\hat{f}(\omega) = \sqrt{2\pi}C(\omega),$$

so that we can write

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt, \quad (4.1)$$

with

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega. \quad (4.2)$$

Equation (4.1) is the definition of the Fourier transform. Equation (4.2) defines the inverse Fourier transform. An alternative notation for $\hat{f}(\omega)$ is $F(f)$.

The Matlab Symbolic Toolbox function which finds Fourier transforms is `fourier` and that which finds the inverse transform is `ifourier`. You should check some of the examples – below in these printed notes – against the results of these Matlab functions.

### 4.3 Examples

**Square pulse** Consider a square pulse of the form

$$f(t) = \begin{cases} 
0 & \text{if } t < 0, \\
k & \text{if } 0 \leq t \leq a, \\
0 & \text{if } a < t.
\end{cases}$$
The Fourier transform is then given by

\[
\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{0}^{a} ke^{-i\omega t} dt
\]

\[
= \frac{k}{\sqrt{2\pi}} \left[ \frac{e^{-i\omega t}}{-i\omega} \right]_{0}^{a} = \frac{k}{i\omega \sqrt{2\pi}} \left( 1 - e^{-i\omega a} \right).
\]

In general, the Fourier transform will be a complex-valued function, i.e., the value of the function for a given \( \omega \) may be complex. The real part of the Fourier transform, i.e. \( \Re(\mathcal{F}(f)) \) is depicted in Fig. 4.1. The imaginary part of the Fourier transform, i.e. \( \Im(\mathcal{F}(f)) \) is plotted in Fig. 4.2. As an exercise, you can verify that the real and imaginary parts of the Fourier transform correspond to the Fourier cosine transform and the Fourier sine transform of the square pulse respectively.

**The Fourier transform of** \( u(t)e^{-t} \) **The Heaviside function** \( u(t - a) \) **is defined as**

\[
u(t - a) = \begin{cases} 
0 & \text{if } t < a, \\
1 & \text{if } t \geq a.
\end{cases}
\]

Therefore, the function \( f(t) = u(t)e^{-t} \) is given by

\[
f(t) = \begin{cases} 
0 & \text{if } t < 0, \\
e^{-t} & \text{if } t \geq 0.
\end{cases}
\]

The Fourier transform of \( f(t) \) is given by

\[
\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} e^{-i\omega t} dt,
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-(1+i\omega)t}}{(1+i\omega)} \right]_{0}^{\infty}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1+i\omega} \right) = \frac{1}{\sqrt{2\pi}} \left( \frac{1 - i\omega}{1 + \omega^2} \right).
\]
Figure 4.3: The real part of the Fourier transform of \( f(t) = u(t)e^{-t} \).

Figure 4.4: The imaginary part of the Fourier transform of \( f(t) = u(t)e^{-t} \).

Notice that
\[
\lim_{t \to \infty} e^{-(1+i\omega)t} = \lim_{t \to \infty} e^{-t}e^{-i\omega t} = \lim_{t \to \infty} e^{-t}(\cos(\omega t) - i\sin(\omega t)) = 0.
\]

Fig. 4.3 depicts the real part of the Fourier transform (4.3). The imaginary part is depicted in Fig. 4.4. Here again, the real and imaginary part of the Fourier transform correspond to the Fourier cosine and Fourier sine transform of \( f(t) = e^{-t} \) respectively. (To take the Fourier cosine transform, we have to create an even extension to \( e^{-t} \), and, for the Fourier sine transform, we need an odd extension of \( e^{-t} \).)

### 4.4 Linearity

If \( f(t) \) and \( g(t) \) are two functions for which the Fourier transform exists, then
\[
\mathcal{F}(af + bg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(t) + bg(t))e^{-i\omega t}dt
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} af(t)e^{-i\omega t}dt + \int_{-\infty}^{\infty} bg(t)e^{-i\omega t}dt \right)
\]
\[
= \left( a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt \right)
\]
\[
= a\mathcal{F}(f) + b\mathcal{F}(g).
\]

We can use this property to calculate the Fourier transform of the function \( f(t) = u(t)e^{-t} + u(t)e^{-2t} \). We have already shown that
\[
\mathcal{F}(u(t)e^{-t}) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + i\omega}.
\]
4.5. SHIFT THEOREMS

In the same way, one finds that

\[
\mathcal{F}(u(t)e^{-2t}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t)e^{-2t}e^{-i\omega t} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(2+i\omega)t} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ \frac{-e^{-(2+i\omega)t}}{2+i\omega} \right]_{0}^{\infty} = \frac{1}{\sqrt{2\pi}} \frac{1}{2+i\omega}.
\]

Using the linearity property, the Fourier transform of \( f(t) = u(t)e^{-t} + u(t)e^{-2t} \) can be found as

\[
\mathcal{F}(u(t)e^{-t} + u(t)e^{-2t}) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega} + \frac{1}{\sqrt{2\pi}} \frac{1}{2+i\omega}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1+i\omega} + \frac{1}{2+i\omega} \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{2+i\omega + 1 + i\omega}{(1+i\omega)(2+i\omega)} \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{3 + 2i\omega}{2 - \omega^2 + 3i\omega} \right)
\]

\[
= \frac{1}{\sqrt{2\pi}} \frac{(3 + 2i\omega)(2 - \omega^2 - 3i\omega)}{(2 - \omega)^2 + 9\omega^2}.
\]

Notice that we have multiplied both the numerator and the denominator with the complex conjugate of the denominator to obtain a real denominator in the final result. This is the standard route to split a complex rational function into a real and imaginary part.

4.5 Shift theorems

4.5.1 The first shift theorem

When \( \mathcal{F}(f) \) is the Fourier transform of a function \( f(t) \) then

\[
\mathcal{F}(e^{iat}f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{iat}e^{-i\omega t} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i(\omega-a)t} dt
\]

\[
= \mathcal{F}(f)(\omega-a).
\]

In summary,

\[
\mathcal{F}(e^{iat}f(t)) = \mathcal{F}(f)(\omega-a),
\]

where \( \mathcal{F}(f)(\omega) \) is the Fourier transform of \( f(t) \).

4.5.2 Applications

You should be able to verify that the Fourier transform of

\[
f(t) = \begin{cases} 
0 & \text{if } t < -2, \\
3 & \text{if } -2 \leq t \leq 2, \\
0 & \text{if } 2 < t.
\end{cases}
\]
Figure 4.5: The Fourier transform of the square pulse $f(t)$.

The Fourier transform $F(f)$ is a real-valued function of $\omega$ and is depicted in Fig. 4.5. The Fourier transform of the function $g(t) = e^{-it}f(t)$

$$g(t) = \begin{cases} 
0 & \text{if } t < -2, \\
3e^{-it} & \text{if } -2 \leq t \leq 2, \\
0 & \text{if } 2 < t.
\end{cases}$$

can be found using the first shift theorem as

$$F(g)(\omega) = \frac{1}{\sqrt{2\pi}} \frac{6 \sin(2(1 + \omega))}{1 + \omega}.$$

Fig. 4.6 gives the Fourier transform of the periodic function $g(t)$. Notice the shift in $\omega$ due to the extra oscillatory factor $e^{it} = \cos(t) + i\sin(t)$. If you multiply a signal with a sinusoidal signal then the Fourier transform of this product is shifted with the frequency corresponding to the sinusoidal signal.

We have seen above that

$$\frac{1}{\sqrt{2\pi}} \frac{1}{2 + i\omega}$$

is the Fourier transform of the function

$$f(t) = u(t)e^{-2t}.$$
Figure 4.7: The Fourier transform of the square pulse \( f(t) \).

The first shift theorem then tells us that

\[
\frac{1}{\sqrt{2\pi}} \frac{1}{2 + i(\omega - 2)}
\]

is the Fourier transform of the function \( g(t) = u(t)e^{-2t}e^{2it} \) which can be written somewhat simpler as \( g(t) = u(t)e^{2i(t-1)t} \).

4.5.3 The second shift theorem

If \( \mathcal{F}(f)(\omega) \) is the Fourier transform of a function \( f(t) \), then

\[
\mathcal{F}(f(t - \alpha)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t - \alpha) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t - \alpha) e^{-i\omega(t-\alpha)} e^{-i\omega\alpha} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} e^{-i\omega\alpha} \int_{-\infty}^{\infty} f(t - \alpha) e^{-i\omega(t-\alpha)} d(t - \alpha)
\]

\[
= \frac{1}{\sqrt{2\pi}} e^{-i\omega\alpha} \int_{-\infty}^{\infty} f(\nu) e^{-i\omega\nu} d\nu
\]

\[
= e^{-i\omega\alpha} \mathcal{F}(f)(\omega).
\]

So that

\[
\mathcal{F}(f(t - \alpha)) = e^{-i\omega\alpha} \mathcal{F}(f)(\omega).
\]

4.5.4 Application

The Fourier transform of

\[
f(t) = \begin{cases} 
0 & \text{if } t < -1, \\
1 & \text{if } -1 \leq t \leq 1, \\
0 & \text{if } 1 < t.
\end{cases}
\]

is

\[
\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \frac{2\sin(\omega)}{\omega}.
\]

This Fourier transform is depicted in Fig. 4.7.

The second shift theorem then indicates that the function

\[
g(t) = \begin{cases} 
0 & \text{if } t < 1, \\
1 & \text{if } 1 \leq t \leq 3, \\
0 & \text{if } 3 < t,
\end{cases}
\]
which can be written as $g(t) = f(t - 2)$, has as Fourier transform

$$
\mathcal{F}(g) = \frac{1}{\sqrt{2\pi}} \frac{2e^{-2i\omega} \sin(\omega)}{\omega}.
$$

$$
= \frac{2}{\sqrt{2\pi}} \frac{\cos(2\omega) \sin(\omega) - i \sin(2\omega) \sin(\omega)}{\omega}
$$

$$
= \frac{1}{\sqrt{2\pi}} \left( \frac{\sin(3\omega) - \sin(\omega)}{\omega} \right) - \frac{i}{\sqrt{2\pi}} \left( \frac{\cos(\omega) - \cos(3\omega)}{\omega} \right).
$$

So the Fourier transform of the shifted function now has frequency components in $\omega$ and $3\omega$. The real and imaginary part of the Fourier transform of the shifted signal are depicted in Fig. 4.8 and Fig. 4.9.

### 4.6 The Fourier transform of $f'(t)$

Take a function $f(t)$ for which the Fourier transform exists and for which $f(t) \to 0$ when $|t| \to \infty$. Then

$$
\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t)e^{-i\omega t} dt
$$

$$
= \frac{1}{\sqrt{2\pi}} \left( \left[ f(t)e^{-i\omega t} \right]_{-\infty}^{\infty} - ( -i\omega ) \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \right)
$$

$$
= i\omega \mathcal{F}(f). \quad (4.4)
$$

When the derivative of $f(t)$ satisfies the same condition, i.e. $f'(t) \to 0$ for $|t| \to \infty$, we can apply formula (4.4) to the second derivative:

$$
\mathcal{F}(f'') = i\omega \mathcal{F}(f').
$$
4.7. THE $T - \omega$ DUALITY

\[
= (i\omega)^2 \mathcal{F}(f)
= -\omega^2 \mathcal{F}(f).
\]

This formula is useful when we want to solve differential equations using Fourier transforms. It can also be used to determine some rather difficult to calculate Fourier transforms. Consider the function

\[f(t) = te^{-t^2},\]

which can be written as

\[f(t) = -\frac{1}{2} (e^{-t^2})'.\]

Its Fourier transform can then be found as

\[\mathcal{F}(f) = -\frac{1}{2} \mathcal{F} \left( (e^{-t^2})' \right) = -\frac{i\omega}{2} \mathcal{F} \left( e^{-t^2} \right).\]

From a standard Fourier transform table, we can learn that

\[\mathcal{F} \left( e^{-t^2} \right) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4},\]

so that the Fourier transform of $f(t)$ is given by

\[\mathcal{F}(f) = -\frac{i\omega}{2\sqrt{2}} e^{-\omega^2/4}.\]

4.7 The $t - \omega$ duality

If we consider the definition of the inverse Fourier transform (4.2)

\[f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f)(\omega) e^{i\omega t} d\omega,\]

we can see that the $\omega$ is the integration variable and can be replaced by any other symbol, e.g.,

\[f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f)(\nu) e^{i\nu t} d\nu.\]

If we then replace $t$ by $-\omega$, we find that

\[f(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f)(\nu) e^{-i\omega \nu} d\nu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f)(t) e^{-i\omega t} dt = \mathcal{F} \left( \mathcal{F}(f)(t) \right).
\]

So, that if $F(\omega) = \mathcal{F}(f)$ is the Fourier transform of $f(t)$, then $f(-\omega)$ is the Fourier transform of $F(t)$.

For example, the Fourier transform of the square pulse

\[f(t) = \begin{cases} 
0 & \text{if } t < 1, \\
1 & \text{if } -1 \leq t \leq 1, \\
0 & \text{if } 1 < t,
\end{cases}\]
is given by the function

\[ F(f) = F(\omega) = 2 \frac{\sin(\omega)}{\omega}. \]

The duality principle then means that the Fourier transform of

\[ g(t) = F(t) = 2 \frac{\sin(t)}{t}, \]

is given by

\[ F(g) = f(-\omega) = f(\omega). \]

### 4.8 Some special Fourier transforms

**The Dirac function \( \delta(t - a) \)** The Dirac function is defined as

\[
\delta(t - a) = \begin{cases} 
0 & \text{if } t < a, \\
1 & \text{if } t = a, \\
0 & \text{if } t > a.
\end{cases}
\]

Its Fourier transform is given by

\[
F(\delta(t - a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t - a)e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} e^{-i\omega a}.
\]

In particular, if \( a = 0 \), we find that

\[ F(\delta(t)) = \frac{1}{\sqrt{2\pi}}. \]

**A constant signal** If we apply the duality principle on the Fourier transform \( F(\omega) = 1/\sqrt{2\pi} \) of \( f(t) = \delta(t) \) we find that

\[ F(1) = \sqrt{2\pi} f(-\omega) = \sqrt{2\pi} \delta(-\omega) = \sqrt{2\pi} \delta(\omega). \]

This means that a constant signal has only one frequency component with frequency \( \omega = 0 \).

**A sinusoidal signal** Applying the duality principle on the Fourier transform of \( \delta(t - a) \), we find that

\[ F(e^{-i\omega t}) = \sqrt{2\pi} f(-\omega) = \sqrt{2\pi} \delta(-\omega - a) = \sqrt{2\pi} \delta(\omega + a). \]

So the Fourier transform of a sinusoidal signal is only different from zero in the frequency of that signal, i.e., a sinusoidal signal has only one frequency component!

**\( \cos(at) \) and \( \sin(at) \)** Since we know that

\[ \cos(at) = \frac{1}{2} \left( e^{iat} + e^{-iat} \right), \]
4.9 Convolution Theorem

The Fourier transform is found as

\[ \mathcal{F}(\cos(at)) = \frac{1}{2} \mathcal{F}(e^{iat} + e^{-iat}) = \sqrt{\frac{2}{\pi}} (\delta(\omega-a) + \delta(\omega+a)) = \frac{\sqrt{\pi}}{2} (\delta(\omega-a) + \delta(\omega+a)) . \]

Similarly, one can find that the Fourier transform of \( \sin(at) \) is given by

\[ \mathcal{F}(\sin(at)) = \sqrt{\frac{\pi}{2}} \frac{1}{i} (\delta(\omega-a) - \delta(\omega+a)) . \]

4.9 Convolution Theorem

Define the Fourier convolution, denoted \( h \) in next equation, of two functions \( f, g \) both defined on \( \mathbb{R} \) by

\[ h(t) = \int_{-\infty}^{\infty} f(u)g(t-u) \, du . \]

It is usual to denote the convolution with some form of star, and here I will use one subscripted with an \( F \):

\[ h(t) = (f *_{F} g)(t) . \]

It is easy to show that convolution is commutative:

\[ (f *_{F} g)(t) = (g *_{F} f)(t) . \]

Harder to show, but easier to remember is the following result concerning Fourier transforms:

\[ \mathcal{F}(f *_{F} g) = (\mathcal{F} g) (\mathcal{F} f) , \]

i.e. the Fourier transform of a Fourier-convolution is the product of the Fourier transforms.

4.10 Numerical computation of Fourier transforms

In practice, it is the exception to be able to find Fourier transforms exactly with closed form formulae. Numerical approximations to Fourier transforms are, in practice, done by “Fast Fourier Transforms”, \texttt{fft}. (E.g. Matlab has a command \texttt{fft}.) A good introductory reference is Chapter 11 of Harman, Dabney and Richert Advanced Engineering Mathematics with Matlab (2nd ed.). As explained in that book, the \texttt{fft} is actually an algorithm for calculating Discrete Fourier Transforms, DFT. DFT computations amount to multiplying by special (unitary, if you want to know) matrices. The DFT has many properties (e.g. a Convolution Theorem) analogous to corresponding properties of the continuous Fourier transform we have just treated. Engineers involved with Signal Processing, or about topics like Time Series (as in all sorts of applications - wave climates for off-shore engineering applications, rainfall and hydrological responses for water-supply engineering) will learn more about it for applications in such engineering units. Just look at Harman, Dabney and Richert if you want some more information on DFT and \texttt{fft}.
4.11 Exercises: Fourier Transforms

1. (a) Find the Fourier transform of the function
\[
f(t) = \begin{cases} 
0 & t < -\pi \\
1 & -\pi \leq t \leq \pi \\
0 & \pi < t 
\end{cases}
\]

(b) and the Fourier transform of
\[
f(t) = \begin{cases} 
0 & t < -2\pi \\
\frac{1}{2} & -2\pi \leq t < -\pi \\
1 & -\pi \leq t \leq \pi \\
\frac{1}{2} & \pi < t \leq 2\pi \\
0 & 2\pi < t 
\end{cases}
\]

2. Find the Fourier transform of the function defined by
\[
f(t) = \begin{cases} 
0 & t < -\pi \\
\pi + t & -\pi \leq t < -\frac{\pi}{2} \\
\frac{\pi}{2} & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\
\pi - t & \frac{\pi}{2} < t \leq \pi \\
0 & \pi < t 
\end{cases}
\]
also sketch the function.

3. (a) Obtain the Fourier transform of the function
\[
f(t) = \begin{cases} 
0 & t < -\frac{\pi}{2} \\
t & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\
0 & \frac{\pi}{2} < t 
\end{cases}
\]
and

(b) the Fourier transform of
\[
f(t) = \begin{cases} 
0 & t < -\frac{\pi}{2} \\
t + \frac{\pi}{2} & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\
0 & \frac{\pi}{2} < t 
\end{cases}
\]

4. Find the Fourier transform of the function
\[
f(t) = \begin{cases} 
e^{-t} & t > 0 \\
e^{t} & t \leq 0 
\end{cases}
\]

5. Sketch the function
\[
f(t) = \begin{cases} 
0 & t < 0 \\
1 & 0 \leq t \leq 1 \\
e^{1-t} & 1 < t 
\end{cases}
\]
and find its Fourier transform.

6. Find the Fourier transform of the square pulse
\[
f(t) = \begin{cases} 
0 & t < b \\
1 & b \leq t \leq 3b \\
0 & 3b < t 
\end{cases}
\]
4.11.1 Solutions: Fourier Transforms

Recall that

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt
\]

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega
\]

1a

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-i\omega t} \, dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \left[ -\frac{1}{i\omega} e^{-i\omega t} \right]_{-\pi}^{\pi}
\]

\[
= \frac{i}{\omega \sqrt{2\pi}} \{ e^{-i\omega \pi} - e^{i\omega \pi} \}
\]

\[
= \frac{i}{\omega \sqrt{2\pi}} (-2i \sin \omega \pi) = \frac{2}{\omega \sqrt{2\pi}} \sin \omega \pi.
\]

1b

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\pi}^{ \frac{\pi}{2} } \frac{1}{2} e^{-i\omega t} \, dt + \int_{ \frac{\pi}{2} }^{ \pi } e^{-i\omega t} \, dt + \int_{\pi}^{\frac{2\pi}{2} } \frac{1}{2} e^{-i\omega t} \, dt \right\}
\]

\[
= \frac{1}{\sqrt{2\pi \omega}} (\sin 2\pi \omega + \sin \pi \omega)
\]

2

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left( \frac{\pi}{2} - \int_{-\pi}^{\pi} (\pi + t)e^{-i\omega t} \, dt + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\frac{\pi}{2}} \frac{1}{2} e^{-i\omega t} \, dt + \frac{1}{\sqrt{2\pi}} \int_{\pi}^{\frac{2\pi}{2} } \frac{1}{2} e^{-i\omega t} \, dt \right)
\]

\[
I_1
\]

\[
\int_{-\pi}^{\frac{\pi}{2} } (\pi + t)e^{-i\omega t} \, dt = \left[ -\frac{\pi + t}{i\omega} e^{-i\omega t} \right]_{-\pi}^{\frac{\pi}{2}} + \frac{1}{i\omega} \int_{-\pi}^{\frac{\pi}{2} } e^{-i\omega t} \, dt
\]
Figure 4.11: Solution to 2

\[
\hat{f}(\omega) = -\frac{\pi}{2i\omega}e^{\frac{i\omega}{2}} + \frac{1}{\omega^2} \left[ e^{\frac{i\omega}{2}} - e^{i\omega\pi} \right]
\]

\(I_2\)

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2} e^{-i\omega t} dt = \frac{\omega}{\omega} \sin \frac{\omega\pi}{2}.
\]

\(I_3\)

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\pi - t) e^{-i\omega t} dt = \left[ -\frac{\pi - t}{i\omega} e^{-i\omega t} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{i\omega} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\omega t} dt
\]

\[
= \frac{\pi}{2i\omega}e^{-\frac{i\pi}{2}} - \frac{1}{\omega^2} \left[ e^{-i\omega\pi} - e^{-\frac{i\pi}{2}} \right]
\]

which can be combined to give

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ -\frac{\pi}{\omega} \sin \frac{\pi\omega}{2} + \frac{2}{\omega} \sin \frac{\omega\pi}{2} + \frac{1}{\omega^2} \left( \cos \frac{\omega\pi}{2} - \cos \omega\pi \right) \right\}
\]

3a

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} te^{-i\omega t} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{t}{i\omega} e^{-i\omega t} \right\}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\omega} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\omega t} dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \left\{ \frac{i\pi}{\omega} \cos \frac{\omega\pi}{2} - \frac{2i}{\omega^2} \sin \frac{\omega\pi}{2} \right\}
\]

3b Use the linearity properties of Fourier transforms, namely \(\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)\) where \(f(t)\) is as in (3a) and

\[
g(t) = \begin{cases} 
0 & 0 < t < \frac{\pi}{2} \\
\frac{\pi}{2} & -\frac{\pi}{2} < t < \frac{\pi}{2} \\
0 & \frac{\pi}{2} < t
\end{cases}
\]
Figure 4.12: The imaginary part of the solution of (3a)

\[ \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{\pi}{\omega} \sin \frac{\omega \pi}{2}. \]

and then the solution is the sum of the two

\[ \frac{1}{\sqrt{2\pi}} \left\{ \frac{i\pi}{\omega} \cos \frac{\omega \pi}{2} - \frac{2i}{\omega^2} \sin \frac{\omega \pi}{2} + \frac{\pi}{\omega} \sin \frac{\omega \pi}{2} \right\} \]

4

\[ \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{t} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-t} e^{-i\omega t} dt \right\} \]
\[ = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{t} e^{-t + i\omega} dt + \int_{0}^{\infty} e^{-t} e^{t + i\omega} dt \right\} \]
\[ = \frac{1}{\sqrt{2\pi}} \left\{ \left[ \frac{1}{1 - i\omega} e^{-t + i\omega} \right]_{-\infty}^{0} + \left[ \frac{1}{1 + i\omega} e^{t + i\omega} \right]_{0}^{\infty} \right\} \]
\[ = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1 - i\omega} + \frac{1}{1 + i\omega} \right) = \frac{1}{\sqrt{2\pi}} \frac{2}{1 + \omega^2} \]

5

\[ \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{0}^{1} e^{-i\omega t} dt + \int_{1}^{\infty} e^{-(1-t)} e^{-i\omega t} dt \right\} \]
\[ = \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{\omega} \sin \omega + \frac{\cos \omega - \omega \sin \omega}{1 + \omega^2} + i \left\{ \frac{\cos \omega - 1}{\omega} - \frac{\omega \cos \omega + \sin \omega}{1 + \omega^2} \right\} \right\} \]

6

\[ \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{3b} e^{-i\omega t} dt \]
\[ = \frac{1}{\omega \sqrt{2\pi}} \{-\sin \omega b + \sin 3\omega b + i(\cos 3\omega b - \cos \omega b)\} \]
Figure 4.13: The real and imaginary parts of the solutions of (3b)

Figure 4.14: The amplitude of the solution of (3b)

Figure 4.15: The argument of the solution of (3b)

Figure 4.16: The amplitude of the solution of (6)
Figure 4.17: The argument of the solution of (6)
Chapter 5

The Fourier Spectrum

We have seen that we can write periodic signals in terms of different frequency components using the theory of Fourier series. In other words, only well specified frequencies contribute to the overall periodic signal. For aperiodic signals, the discrete nature of the summation in the Fourier series is replaced by the continuous nature of the Fourier integral. All frequencies do contribute to the aperiodic signal. We do no longer have coefficients for well specified frequencies only, but a coefficient which is a function of the frequency. We have renamed this coefficient as the Fourier transform. It tells us how different frequencies contribute to the aperiodic signal.

The representation of a signal as a combination (either a sum or an integral) of sinusoidal contributions of different frequencies is called a spectral representation. Periodic signals have a discrete spectrum or a point spectrum, aperiodic signals have a continuous spectrum.

5.1 The Fourier (frequency) spectrum

If \( F(\omega) \) is the Fourier transform of a signal \( f(t) \) then \( F(\omega) \) is also called the complex frequency spectrum of the signal \( f(t) \). We have seen that the Fourier transform is in general a complex-valued function of \( \omega \). We have investigated the Fourier frequency spectrum in the previous chapter by looking at the real and imaginary parts of the Fourier transform. A more common way of looking at a complex function is by using the exponential notation for a complex number.

The complex number \( z = z_r + iz_i \) can be written in exponential notation as

\[
z = r (\cos(\phi) + i \sin(\phi)) = re^{i\phi},
\]

with \( r = |z| \) the modulus of \( z \),

\[
|z| = \sqrt{zz^*}.
\]

Here, \( z^* \) is the complex conjugate of \( z \), i.e., \( z^* = z_r - iz_i \). The phase \( \phi \) is given by

\[
\phi = \tan^{-1}\left(\frac{z_i}{z_r}\right).
\]

In the same way, we can write the complex frequency spectrum as

\[
F(\omega) = |F(\omega)| e^{i\Phi},
\]

with

\[
\Phi = \arg(F(\omega)) = \tan^{-1}\left(\frac{\Im(F(\omega))}{\Re(F(\omega))}\right).
\]
Figure 5.1: The amplitude spectrum for the signal $u(t)e^{-at}$ with $a = 3$.

Figure 5.2: The phase spectrum for the signal $u(t)e^{-at}$ with $a = 3$.

Both $|F(\omega)|$ and $\arg(F(\omega))$ are real functions of $\omega$. $|F(\omega)|$ is called the amplitude spectrum. It tells us which frequencies contribute most to the signal. The argument of $F(\omega)$, $\arg(F(\omega))$, is the phase spectrum and is related to the phase of a particular frequency contribution.

5.2 Examples

$f(t) = u(t)e^{-at}$ The Fourier transform of $u(t)e^{-at}$ can easily be found as

$$F(f) = F(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{a + i\omega} = \frac{1}{\sqrt{2\pi}} \frac{a - i\omega}{a^2 + \omega^2}.$$  

The amplitude spectrum is then given by

$$|F(\omega)| = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a - i\omega}{a^2 + \omega^2}} \left(\frac{a + i\omega}{a^2 + \omega^2}\right) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a^2 + \omega^2}}. \quad (5.1)$$

Fig. 5.1 pictures the amplitude spectrum (5.1) for $a = 3$. Notice that it peaks at zero frequency and that it drops to zero for $\omega \to \infty$ and for $\omega \to -\infty$.

The phase spectrum is given by

$$\arg(F(\omega)) = \tan^{-1} \left(\frac{1}{\sqrt{2\pi}} \left(\frac{-\omega}{a^2 + \omega^2}\right) \sqrt{2\pi} \left(\frac{a^2 + \omega^2}{a}\right)\right)$$

$$= \tan^{-1} \left(-\frac{\omega}{a}\right) = -\tan^{-1} \left(\frac{\omega}{a}\right).$$

The phase spectrum is depicted in Fig 5.2.
The square pulse  The square pulse

\[ g(t) = \begin{cases} 
0 & \text{if } t < -1, \\
1 & \text{if } -1 \leq t \leq 1, \\
0 & \text{if } 1 < t.
\end{cases} \]

has a Fourier transform

\[ \mathcal{F}(g) = G(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega}. \]

\( G(\omega) \) is a real valued function, and, therefore, the amplitude spectrum is given by the absolute value of \( G(\omega) \). It is plotted in Fig. 5.3. The phase spectrum is given by

\[ \arg(G(\omega)) = \tan^{-1}(0) = 0 \text{ or } \pi. \]

For a positive amplitude, the phase is zero, while for a negative amplitude, the phase must be \( \pi \), i.e.,

\[ \arg(G(\omega)) = \begin{cases} 
0 & \text{if } \frac{\sin(\omega)}{\omega} \geq 0, \\
\pi & \text{if } \frac{\sin(\omega)}{\omega} < 0.
\end{cases} \]

The amplitude spectrum and the phase spectrum are given in Fig. 5.3 and Fig. 5.4 respectively. Again, the amplitude spectrum drops off to zero for large (positive or negative) frequencies.
5.3 The energy spectral density

The total energy contained in a signal is defined as

$$E = \int_{-\infty}^{\infty} (f(t))^2 \, dt.$$ 

If $F(\omega)$ is the Fourier transform of the signal $f(t)$ then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} \, d\omega,$$

and we can express the energy as

$$E = \int_{-\infty}^{\infty} f(t)f(t) dt = \int_{-\infty}^{\infty} f(t) \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} \, d\omega \right) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \left( \int_{-\infty}^{\infty} f(t)e^{i\omega t} \, dt \right) d\omega = \int_{-\infty}^{\infty} F(\omega)F^*(\omega) d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega.$$ 

$F^*(\omega)$ is the complex conjugate of $F(\omega)$. Therefore, the amplitude spectrum contains information about the distribution of energy over the different frequencies. We can calculate the total energy in terms of the signal or in terms of its Fourier transform as

$$E = \int_{-\infty}^{\infty} (f(t))^2 \, dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega. \quad (5.2)$$

The relation (5.2) is also known as Parseval’s Theorem, and, $|F(\omega)|^2$ is often called the energy spectral density.

5.4 Periodic signal

A periodic signal $f(t)$ with period $p = 2L$ can be written as a (complex) Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} = \sum_{n=-\infty}^{\infty} c_n e^{i\pi n t},$$

with $\omega_n = n\pi/L$. The Fourier transform of $f(t)$ can then be calculated as

$$F(f) = F\left( \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \right) = \sum_{n=-\infty}^{\infty} c_n F(e^{i\omega_n t}) = \sum_{n=-\infty}^{\infty} c_n \sqrt{2\pi} \delta(\omega - \omega_n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} c_n \delta(\omega - \omega_n).$$

The Fourier spectrum of a periodic signal is thus a discrete spectrum.
5.5 Amplitude modulation

Consider the amplitude modulated signal

\[ g(t) = x(t) \cos(\omega_c t), \]

where \( \omega_c \) is the (angular) frequency of the carrier signal and \( x(t) \) is the modulation signal. The Fourier transform of the amplitude modulated signal can be calculated in the following way:

\[
\mathcal{F}(g) = \mathcal{F}(x(t) \cos(\omega_c t)) \\
= \mathcal{F} \left( \frac{x(t)e^{i\omega_c t} + x(t)e^{-i\omega_c t}}{2} \right) \\
= \mathcal{F} \left( \frac{x(t)e^{i\omega_c t}}{2} \right) + \mathcal{F} \left( \frac{x(t)e^{-i\omega_c t}}{2} \right) \\
= \frac{1}{2} \mathcal{F}(x(t)e^{i\omega_c t}) + \frac{1}{2} \mathcal{F}(x(t)e^{-i\omega_c t}) \\
= \frac{1}{2} X(\omega - \omega_c) + \frac{1}{2} X(\omega + \omega_c),
\]

where \( X(\omega) \) is the Fourier transform of \( x(t) \).

Figure 5.5: The amplitude spectrum of the amplitude modulated signal with \( \omega_c = 3\pi \).

When we consider the modulation signal \( x(t) = u(t)e^{-t} \) with Fourier transform

\[
X(t) = \frac{1}{\sqrt{2\pi}} \frac{1 - i\omega}{1 + \omega^2},
\]

the Fourier transform of the amplitude modulated signal is given by

\[
\mathcal{F}(g) = \frac{1}{2\sqrt{2\pi}} \left( \frac{1 - i(\omega - \omega_c)}{1 + (\omega - \omega_c)^2} + \frac{1 - i(\omega + \omega_c)}{1 + (\omega + \omega_c)^2} \right).
\]

The amplitude spectrum of the amplitude modulated signal is depicted in Fig. 5.5. One advantage of the higher frequencies in the amplitude modulated spectrum is that one can use a smaller antenna to transmit it.
Chapter 6

Further topics – fft

6.1 The Discrete Fourier Transform and the FFT

See [StrBC] §3.5 and §5.5 p299, or [StrWC] §10.3.

The FFT is an efficient method of computing discrete Fourier transforms, DFTs.

Definition. The \textit{discrete Fourier transform} (DFT) of an \(N\)-tuple of complex numbers \((f(kT))\) \((k = 0, \ldots, N - 1)\) is the following \(N\)-tuple:

\[
F \left( \frac{n}{NT} \right) = \sum_{k=0}^{N-1} f(kT) \exp(-2\pi i kn/N) \quad \text{for } n = 0, 1, \ldots, N - 1.
\]

The \textit{inverse discrete Fourier transform} (IDFT) of an \(N\)-tuple of complex numbers \((F(n/(NT)))\) \((n = 0, \ldots, N - 1)\) is the following \(N\)-tuple:

\[
f(kT) = \sum_{n=0}^{N-1} F \left( \frac{n}{NT} \right) \exp(2\pi i kn/N) \quad \text{for } k = 0, 1, \ldots, N - 1.
\]

Note that these transforms are just matrix multiplications. For example, with \(N = 4\) the matrix is, with \(\omega_4 = \exp(2\pi i/4) = i\),

\[
F_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega_4 & \omega_4^2 & \omega_4^3 \\
1 & \omega_4^2 & \omega_4^3 & \omega_4 \\
1 & \omega_4^3 & \omega_4 & \omega_4^2
\end{bmatrix}
\]

The \(N\) by \(N\) matrix \(F_N\), defined using \(\omega_N = \exp(2\pi i/N)\), is a symmetric van-der-Monde matrix and \(\frac{1}{\sqrt{N}}F_N\) is unitary.

The DFT can be used as a numerical approximation method for ordinary Fourier transforms. As an example, consider \(f(t) = \exp(-t)\text{Heaviside}(t)\): (up to multiples of \(\sqrt{2}\pi\) depending on definition used) the Fourier transform is

\[
F(\omega) = \frac{1}{1 + \omega^2} - \frac{\omega}{1 + \omega^2} i.
\]

Many theorems are available in similar forms for both discrete and continuous transforms.

Definition. Suppose the \(a\) and \(b\) are vectors in \(\mathbb{C}^N\). The \textit{cyclic-convolution of \(a\) and \(b\)} is defined as the \(N\)-vector whose \(k\)-th entry is given as below

\[
\text{cyclic\_conv}(a, b) = \left[ \sum_{j=0}^{N-1} a_j b_{k-j} \right] \quad \text{with index } k = 0, \ldots, (N - 1),
\]
where negative indices in the summation are taken mod $N$.

**EXAMPLE.** $N = 2$:

\[
\text{cyclic\_conv}([a_0, a_1], [b_0, b_1]) = [a_0 b_0 + a_1 b_1, a_0 b_1 + a_1 b_0].
\]

**THEOREM**

\[
\text{DFT} (\text{cyclic\_conv}(a, b)) = \text{componentwise product} (\text{DFT}(a), \text{DFT}(b)).
\]

Caution. With some definitions of DFT a factor of $\sqrt{N}$ might enter in the statement of the Convolution Theorem.

Calculation of DFTs amounts to matrix multiplication. A very efficient method of calculating them comes from a very special factorisation (\cite{StrBC} §3.5 or \cite{StrWC} §10.3) of the matrices $F_N$:

\[
F_{2N} = \begin{bmatrix} I_N & D_N \\ I_N & -D_N \end{bmatrix} \begin{bmatrix} F_N & 0 \\ 0 & F_N \end{bmatrix} P_{2N},
\]

where $D_N$ is the diagonal matrix with entries $1, \omega_{2N}, \ldots, \omega_{2N}^{N-1}$ along the diagonal, and $P_{2N}$ is the permutation matrix which separates the incoming vector into first its even subscripted parts, then its odd parts.

The basic idea is to convert a multiplication by a full matrix into a product of a modest number of very sparse, very structured matrices, and thereby save computational effort.
Bibliography


