EIGENVALUES OF GENERAL (SQUARE) MATRICES
AND LINEAR SYSTEMS OF DEs

1 Eigenvalues

1.1 Eigenvectors and systems $y' = Ay$

Consider

$$\frac{dy}{dt} = Ay,$$

where $A$ is a square matrix. If we look for solutions $y = u \exp(\lambda t)$, on substituting into the d.e., we see that, to get a solution, we must have

$$\lambda u = Au \quad \text{or equivalently} \quad (\lambda I - A)u = 0.$$ 

A nonzero vector $u$ which satisfies this last equation is called an eigenvector, and the number $\lambda$ is called an eigenvalue. We will return to how best to use this sort of information to solve d.e.s soon.

We warn students that the order of presentation in the lectures will not follow that of the notes exactly. In particular, in lectures we will be treating the eigenproperties of real symmetric matrices, and their uses in normal modes of oscillation earlier than in these notes. The more elaborate material on Jordan forms will be later in the lectures, and some topics here (the mixing example of a system of d.e.s) will be left for you to read.

1.2 Some definitions and results

The polynomial

$$p_A(t) = \det(tI - A)$$

is called the characteristic polynomial of $A$. $\lambda$ is an eigenvalue if and only if it satisfies $p_A(\lambda) = 0$.

**Definition.** StrBC p251. The trace of $A$, trace($A$) = $\sum_{i=1}^{n} a_{ii}$.

**Theorem.** $p_A(t) = t^n - (\text{trace}(A))t^{n-1} + \ldots + (-1)^n \det(A)$.

For hand-calculation with small matrices $A$ the usual process is:

- form the characteristic polynomial;
- find its roots, which gives us the eigenvalues;
- for each eigenvalue $\lambda$, find a basis for the nullspace of $A - \lambda I$ (using your first year Gaussian elimination, rref, methods).
You should be aware of the following Mathematica functions:

- CharacteristicPolynomial
- Eigensystem
- JordanDecomposition

Small caveat: You – or Mathematica – might need to find nullspaces for matrices with complex number entries. Real matrices can very easily have complex eigenvalues.

Complex eigenvalue example

Let

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]

We have \( p_A(t) = \det(tI_n - A) = t^2 + 1 \) so the eigenvalues \( \lambda_{\pm} \) of \( A \) are +i and −i. The nullspaces of \( A - \lambda I \) are readily found, and give

\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}
\]

There is some geometry associated with this example. \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) represents, as an operator on \( \mathbb{R}^2 \), a rotation through \( \pi/2 \). There is no real direction \( u \) left unchanged by this rotation, hence no real eigenvalues.

**Theorem.** StrBC p250. The eigenvalues of an upper-triangular, lower-triangular, or diagonal matrix are the main diagonal entries.

There are various other useful and memorable facts.

- StrBC 5B. The product of the eigenvalues of \( A \) (included to the power of their algebraic multiplicity, see below for definition) is equal to \( \det(A) \).
- The sum of the eigenvalues of \( A \) (included to the power of their algebraic multiplicity, see below for definition) is equal to \( \text{trace}(A) \).
- \( \text{trace}(AB) = \text{trace}(BA) \).
- The eigenvalues of \( BA \) are the same as the eigenvalues of \( AB \).

### 1.3 Algebraic and Geometric multiplicities

**Definition.** (StrBC p255.) Let \( \lambda \) be an eigenvalue of \( A \). The **algebraic multiplicity** of \( \lambda \) is the largest power \( m \) such that

\[(t - \lambda)^m \text{ divides exactly into } p_A(t).\]

**Definition.** (StrBC p255.) The **geometric multiplicity** of eigenvalue \( \lambda \) is

\[\dim(\{X | (\lambda I - A)X = 0\})\].
THEOREM. The geometric multiplicity of any eigenvalue is less than or equal to its algebraic multiplicity.

Definition. (StrBC p255.) A matrix $A$ is defective (or deficient) if $A$ has an eigenvalue whose geometric multiplicity is (strictly) less than its algebraic multiplicity.

EXAMPLES

- $\begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$ eigenvalues $d_{11}$, $d_{22}$.
- $\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$ eigenvalue $d$ algebraic multiplicity $= 2 = $ geometric multiplicity.
- $\begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$ eigenvalue $d$ algebraic multiplicity $= 2 > $ geometric multiplicity $= 1$.

EXAMPLE. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$ 

Find the eigenvalues, bases for the eigenspaces and the algebraic and geometric multiplicities.

Solution. The matrix $A$ is triangular, so the eigenvalues are $1$, $1$ and $-2$. The eigenvalue $1$ has algebraic multiplicity $2$: the eigenvalue $-2$ has algebraic multiplicity $1$.

A basis for the eigenspace of eigenvalue $1$ is $\{(1, 0, 0)^T\}$ so the geometric multiplicity of eigenvalue $1$ is $1$.

A basis for the eigenspace of eigenvalue $-2$ is $\{(1, 3, -9)^T\}$ and the geometric multiplicity of eigenvalue $-2$ is $1$. (The geometric multiplicity has to be at least one, and it can’t be greater than the algebraic multiplicity.)

1.4 Matrices and similarity

Definition. ([StrBC] §5.6 p304, [StrWC] §6.6 p295.) If $A$ and $B$ are $n$ by $n$ matrices, we say that $B$ is similar to $A$ if and only if $B = P^{-1}AP$ for some nonsingular matrix $P$.

THEOREM. ([StrBC] §5.6 5P, [StrWC] §6.6 Thm 6Q.) Similar matrices have the same characteristic polynomial, the same eigenvalues, the same trace, and the same determinant.

Proof. Let $B = P^{-1}AP$. Then $tI - B = P^{-1}(tI - A)P$. Thus

$$\det(tI - B) = \det(P^{-1}) \det(tI - A) \det(P) = \det(tI - A),$$

so the characteristic polynomials are same.

The case $t = 0$ is that the determinants are same.

The trace result follows from the fact that for any square matrices of the same size $\text{trace}(MN) = \text{trace}(NM)$. Thus,

$$\text{trace}(B) = \text{trace}(P^{-1}AP) = \text{trace}(PP^{-1}A) = \text{trace}(A).$$
THEOREM. Let $A$, $B$, $C$ denote $n$ by $n$ matrices. Then

1. (Reflexive Property.) Every square matrix is similar to itself.
2. (Symmetric Property.) If $A$ is similar to $B$, then $B$ is similar to $A$.
3. (Transitive Property.) If $A$ is similar to $B$, and $B$ is similar to $C$, then $A$ is similar to $C$.

EXAMPLE.

$$D_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \text{ is similar to } D_2 = \begin{bmatrix} d_2 & 0 \\ 0 & d_1 \end{bmatrix}.$$ 

The reason for this is as follows. Let $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $M$ is a permutation matrix and $M^{-1} = M$.

$$M^{-1}D_1M = \begin{bmatrix} 0 & d_2 \\ d_1 & 0 \end{bmatrix} M = \begin{bmatrix} d_2 & 0 \\ 0 & d_1 \end{bmatrix} = D_2,$$

as required.

Here is a result which doesn’t actually mention similarity, but whose proof (which we won’t do in the 2nd year) would belong near here. Actually, it is easy to prove for 2 by 2 matrices by direct calculation (and with a CAS you could check with 3 by 3 matrices too). We deliberately state the result informally, in an easy-to-remember form.

THEOREM. (Cayley-Hamilton Theorem, ([StrBC] Q5.6.24 p317, [StrWC] §6.6 Thm 6Q.) A square matrix satisfies its own characteristic equation.

1.5 Diagonalization

[StrBC] §5.6, [StrWC] §6.2

PROBLEM A1. Given a linear operator $T : V \to V$ on a finite dimensional vector space $V$, does there exist a basis for $V$ w.r.t. which the matrix of $T$ is diagonal?

PROBLEM A2. Given a linear operator $T : V \to V$ on a finite dimensional inner product space $V$ (= a Euclidean space), does there exist an orthonormal basis for $V$ w.r.t. which the matrix of $T$ is diagonal? (Well, the definition of orthonormal is a bit later in these notes, but will be advertised in lectures here.)

Matrix forms of these questions.

PROBLEM M1. Given a square matrix $A$, does there exist an invertible matrix $P$ such that $P^{-1}AP$ is diagonal?

PROBLEM M2. Given a square matrix $A$, does there exist an orthogonal matrix $Q$ such that $Q^T AQ$ is diagonal. (Orthogonal means $Q^T = Q^{-1}$.)

In these notes we concentrate on the matrix problems M1 and M2 rather than the abstract problems A1 and A2. We will consider Problem 2 later. Here, however, is a preview - the answer to Problem M2.

Let $A$ be an $n$ by $n$ real matrix. Then $A$ is orthogonally similar to a diagonal matrix (i.e. $Q^T AQ$ is diagonal) if and only if $A$ is symmetric.

Definition. Matrix $A$ is said to be diagonal, or diagonalizable, if there is an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.
Answer to Problem M1.

**THEOREM.** A is diagonalizable if and only if there exists a basis of \( C^n \) in which every vector is an eigenvector of \( A \), if and only if for any eigenvalue \( \lambda \) of \( A \), the algebraic and geometric multiplicities of \( \lambda \) are equal.

If the answer to Problem M1 is that for a particular \( A \) there is an invertible matrix \( P \) such that \( P^{-1}AP \) is diagonal this leads to further questions. How is \( P \) found?

What can be said about the diagonal matrix \( D = P^{-1}AP \)?

**THEOREM.** If \( A \) is similar to the diagonal matrix \( D = \text{diag}(\alpha_1, \ldots, \alpha_n) \), then \( \alpha_1, \ldots, \alpha_n \) are the eigenvalues of \( A \) (with their algebraic multiplicities).

**THEOREM.** Suppose that \( A \) is diagonalizable, so that by earlier results, there is a basis in which every vector is an eigenvector of \( A \). Let \((X_1, \ldots, X_n)\) be any such basis, with \( AX_k = \alpha_k X_k \), and let
\[
P = [X_1; \ldots; X_n].
\]
Then \( P \) is nonsingular and \( P^{-1}AP = \text{diag}(\alpha_1, \ldots, \alpha_n) \).

For a proof, see [StrBC] Thm 5C.

**THEOREM.** ([StrBC] Thm 5D.) Let \( A \) be an \( n \) by \( n \) matrix. Then eigenvectors corresponding to distinct eigenvalues are linearly independent.

**COROLLARY.** ([StrBC] bottom p256.) An \( n \) by \( n \) matrix with \( n \) distinct eigenvalues is diagonalizable.

If a matrix is diagonalizable, this fact can be useful in calculations. For example, if \( A \) is diagonalizable, \( D = P^{-1}AP \), then any power of \( A \) is easily found, even very large powers, as \( A^k = PD^kP^{-1} \) and \( D^k \) is easily found. ([StrBC] p258.)

Another interesting fact is

([StrBC] Thm 5F.) If \( A \) and \( B \) are diagonalizable, they share the same eigenvector matrix \( P \) if and only if \( AB = BA \).

Although it is easy to find float numeric approximations to eigenvalues, it isn’t easy, in general, to quickly look at a matrix and say anything useful about its eigenvalues. However, there are some special cases when something can be said. Here is an example.

([StrBC] Thm 5K p271.) Let \( A \) be a square matrix, all of whose entries are positive numbers. Then, the eigenvalue \( \lambda_1 \) of \( A \) which is of largest magnitude is (real and) positive, and also all the components of an eigenvector corresponding to \( \lambda_1 \) are (real and) positive.

**1.6 More about similarity: Jordan form**

(It is Camille Jordan, a different Jordan than the one with the Gauss-Jordan decomposition, rref.)

Not all matrices are diagonalizable, e.g. \[
\begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}.
\]

This leads to the following question. What simple form can any \( n \) by \( n \) matrix be reduced to by similarity? I.e., given \( A \), find a simple form for \( B = P^{-1}AP \)?

Answer (but the proof is too advanced to include here):

**JORDAN FORM THEOREM.** ([StrBC] Thm 5U, [StrWC] Thm 6R.) An \( n \) by \( n \) matrix with eigenvalues, repeated with their algebraic multiplicities \((\lambda_1, \ldots, \lambda_n)\) is similar to a matrix with these eigenvalues in the diagonal positions, zeros and ones on the superdiagonal, and zeros elsewhere.
EXAMPLE #different eigenvalues #(lin indep) eigenvectors
\[
\begin{bmatrix}
\lambda & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \nu
\end{bmatrix}
\]
3 3
\[
\begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \mu
\end{bmatrix}
\]
2 3
\[
\begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \mu
\end{bmatrix}
\]
2 2
\[
\begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\]
1 3
\[
\begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\]
1 2
\[
\begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix}
\]
1 1

We will see, in a later section of this set of notes, a USE of jordan forms in connection with solving constant coefficient systems of d.e.s - matrix exponentials.

There is an important caution concerning jordan forms, namely that in Float computation, the process of forming them is very badly conditioned.

EXAMPLE. Let \( A_\epsilon = \begin{bmatrix} \epsilon & 1 \\ 0 & 0 \end{bmatrix} \). When \( \epsilon = 0 \), i.e. \( A_0 \), is already in jordan form. However, for any \( \epsilon \neq 0 \) it, \( A_\epsilon \), is not in jordan form, and \( A_\epsilon \) is diagonalable. Verify this last statement.

Solution. We have
\[
\begin{bmatrix} \epsilon & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \epsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \epsilon & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
In particular, when \( \epsilon \) is small but nonzero we have ‘nearly parallel’ eigenvectors. For the diagonalizing matrix \( P_\epsilon \) we have
\[
P_\epsilon^{-1} = \frac{1}{\epsilon} \begin{bmatrix} \epsilon & 1 \\ 0 & -1 \end{bmatrix}, \quad P_\epsilon = \begin{bmatrix} 1 & 1 \\ 0 & -\epsilon \end{bmatrix}.
\]
2 SYSTEMS OF LINEAR D.E.S: Intro to matrix exponentials, etc.

References: [StrBC] §5.4, [StrWC] §6.3

2.1 Example: compartment models involving solute and mixing

There are huge numbers of examples of linear systems of d.e.s, and you will encounter many in your engineering units. Since we are about to treat constant coefficient systems it may be appropriate to begin with an easy-to-derive example of one that arises in connection with mixing of solutes in a system of interconnected tanks. (It is possible to imagine ‘environmental engineering/environmental science’ applications of a related kind, e.g. concerning salinity levels in different parts of the Swan/Canning river/estuary system, treated as a function of time with given inputs via rainfall, streamflow, and mixing rates between different parts. However, we will treat merely an idealized invented system.)

2.1.1 Two tank example: formulating the d.e.s

Consider two interconnected tanks each containing water and a solute. Both tanks are kept well mixed.

- Initially, tank 1 contains $v_1$ litres of water and $q_1(0)$ kg of solute, and tank 2 contains $v_2$ litres of water and $q_2(0)$ kg of solute.

- Water containing $g_1$ kg of solute per litre flows into tank 1 at a rate of $s_1$ litres per minute. The mixture in tank 1 flows out to tank 2 at a rate of $r_{12}$ litres per minute. The total volume in tank 1 remains constant, with an overflow of $d_1$ litres per minute going into a drain. Water containing $g_2$ kg of solute per litre flows into tank 2 at a rate of $s_2$ litres per minute. The mixture in tank 2 flows out to tank 1 at a rate of $r_{21}$ litres per minute. The total volume in tank 2 remains constant, with an overflow of $d_2$ litres per minute going into a drain.

Figure 1 is a diagram of the situation.

Let $q_i(t)$ denote the quantity of solute in tank $i$ at time $t$. A simple mass balance gives that $q_1(t)$ and $q_2(t)$ satisfy the following system of d.e.s:

\[
\begin{align*}
\frac{d_1}{dt} &= s_1 - r_{12} + r_{21} \\
\frac{d_2}{dt} &= s_2 + r_{12} - r_{21} \\
\frac{dq_1}{dt} &= s_1 g_1 - \frac{(r_{12} + d_1)q_1}{v_1} + \frac{r_{21}q_2}{v_2} \\
\frac{dq_2}{dt} &= s_2 g_2 + \frac{r_{12}q_1}{v_1} - \frac{(r_{21} + d_2)q_2}{v_2}
\end{align*}
\]

It is a bit messy keeping all these symbols general, so let's consider a couple of numerical examples

- the first example being from the (1st year level) book by Fulford et al. *Modelling with differential and difference equations*
• the second example as in the CODEE ODE Architect book published by Wiley (Exploration 6.3, p109), leaving $g_1$, $g_2$ general.

**Fulford example, p356**

Assume that initially the two tanks each contain 100 litres of pure water. Pure dye is then pumped into the first tank at a fixed rate of 1 litre/minute, while pure water is pumped into the second tank at the same rate. Pumps exchange the mixtures between the two tanks, – at a rate of 4 litres/minute from tank 1 to tank 2, and at a rate of 3 litres/minute from tank 2 to tank 1. The diluted mixture is drawn off from tank 2 at a rate of 2 litres/minute. Derive the differential equations for the amount of dye in each tank.

\[
\begin{align*}
v_1 &= 100, \quad q_1(0) = 0 \\
v_2 &= 100, \quad q_2(0) = 0 \\
s_1 &= 1, \quad g_1 = 1, \quad r_{12} = 4, \quad d_1 = 0 \\
s_2 &= 1, \quad g_2 = 1, \quad r_{21} = 3, \quad d_2 = 2
\end{align*}
\]

This leads to the system of d.e.s

\[
\begin{align*}
\frac{dq_1}{dt} &= 1 - \frac{4}{100} q_1 + \frac{3}{100} q_2 \\
\frac{dq_2}{dt} &= 0 + \frac{4}{100} q_1 - \frac{5}{100} q_2
\end{align*}
\]

This can be written as a system

\[
\frac{d\mathbf{q}}{dt} = \mathbf{b} + \mathbf{Aq}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \frac{1}{100} \begin{bmatrix} -4 & 3 \\ 4 & -5 \end{bmatrix}
\]

The mixing problem we have been given is an ‘initial-value problem’: the system of d.e.s is to be solved for $\mathbf{q}(t)$, with $\mathbf{q}(0)$ given.
In this example, \( v_1 = 5, \quad q_1(0) = 3 \)
\( v_2 = 4, \quad q_1(0) = 5 \)
\( s_1 = 2, \quad r_{12} = 3, \quad d_1 = 3 \)
\( s_2 = 3, \quad r_{21} = 4, \quad d_2 = 2 \)

This leads to the system of d.e.s

\[
\frac{dq_1}{dt} = 2g_1 - \frac{6}{5}q_1 + \frac{4}{4}q_2
\]
\[
\frac{dq_2}{dt} = 3g_2 + \frac{3}{5}q_1 - \frac{6}{4}q_2
\]

This can be written as a system

\[
\frac{dq}{dt} = b + Aq, \quad b = \begin{bmatrix} 2g_1 \\ 3g_2 \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{6}{5} & 1 \\ \frac{3}{5} & -\frac{3}{2} \end{bmatrix}
\]

Again, the mixing problem in this example is an ‘initial-value problem’: the system of d.e.s is to be solved for \( q(t) \), with \( q(0) \) given.

The general case

It is no more difficult to set up the d.e.s for the general case. This can be written as a system

\[
\frac{dq}{dt} = b + Aq, \quad b = \begin{bmatrix} s_1g_1 \\ s_2g_2 \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{s_1+r_{21}}{v_1} & \frac{r_{21}}{v_2} \\ \frac{r_{12}}{v_1} & -\frac{s_2+r_{12}}{v_2} \end{bmatrix}
\]

2.1.2 Two tank example: discussing solving the d.e.s

The approach to the problem is much as with your first year linear d.e. work. There is an easily spotted particular solution, an equilibrium solution where \( q(t) \) stays constant, at \( q_e \) where

\[
Aq_e = -b
\]

(Note that, in our numerical examples, \( \det(A) \neq 0 \) so there is a unique equilibrium solution.) We will see that the easiest way to write the solution is

\[
q(t) = q_e + \expm(tA)(q(0) - q_e)
\]

Here \( \expm(At) \) is called the ‘exponential-matrix’ of \( At \).

The numerical versions of the examples here do have one feature which simplifies their solution: for them the matrices \( A \) are diagonalizable. Let \( u_j \) denote an eigenvector corresponding to eigenvalue \( \lambda_j \). From the discussion at the beginning of this chapter on eigenvalues we know that the general solution of the d.e. problem is

\[
q(t) = q_e + c_1u_1\exp(\lambda_1t) + c_2u_2\exp(\lambda_2t)
\]

for constants \( c_1, c_2 \). The initial-value problem is solved by finding the \( c_1, c_2 \) so that the initial conditions are satisfied. There is a lot of tedious, but routine, calculation in this (and it gets even
worse with matrices which are not diagonalizable). Because initial-value problems occur a lot, there
is value in tidy formula like that involving the exponential matrix above.

**Fulford example**

In the Fulford example, \( A \) has eigenvalues of \(-1/100\) and \(-8/100\). The formula for the exponential matrix is just given here (use matlab’s \texttt{expm} if you want to check, but, at this stage I haven’t told you how to calculate \( \text{expm}(tA) \)), and, to save writing, let

\[
e_1 = \exp(-t/100), \quad e_2 = \exp(-8t/100);
\]

\[
\text{expm}(tA) = \frac{1}{7} \begin{bmatrix} 4e_1 + 3e_2 & 3e_1 - 3e_2 \\ 4e_1 - 4e_2 & 3e_1 + 4e_2 \end{bmatrix}
\]

Note that this tends to zero as \( t \) tends to plus infinity, so that any solution – no matter what the starting values are – tends to the equilibrium solution. With this formula for \( \text{expm}(tA) \) and

\[
q(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad q_e = \begin{bmatrix} 125/2 \\ 50 \end{bmatrix},
\]

we can easily plot out the solutions and the plot is shown in Figure 2.

![Figure 2: The quantity of dye in each tank in the Fulford example. The plot is the same as in the Fulford et al. book, p365.](image)

**CODEE example**

In the CODEE example, its \( A \) has pretty messy eigenvalues, and as a consequence the formula for the exponential matrix is a mess. This is pretty typical of real problems, I’m afraid. Actually, at least it is possible to find the eigenvalues. Doing this numerically is often enough, and, in particular,
with this problem it is easily seen that both eigenvalues are negative, and this has the implication that as \( t \) tends to plus infinity, the \( \expm(tA) \) term tends to zero, and so any solution tends to \( q_e \).

At this stage you can ignore the details of this (and, it is recommended that you don’t, on return to this messy problem, work it out by hand, as matlab is a better tool if you must look at the details). I find that

\[
\expm(tA) = \exp(-27t/20) \begin{bmatrix}
  c - \frac{\sqrt{249}}{83} s & \frac{20}{\sqrt{249}} s \\
  4\sqrt{249} s & c - \frac{\sqrt{249}}{83} s
\end{bmatrix}
\]

where \( c = \cosh(t\sqrt{249}/20) \) and \( s = \sinh(t\sqrt{249}/20) \).

In exercise 2, page 109 of the CODEE book, they ask people to plot the solutions when also \( g_1 = 1 \) and \( g_2 = 1 \). The CODEE code is just numerical solution, effectively equivalent to matlab’s \texttt{ode45}, and then it becomes an exploration to see things, easy for us with the formula, that the equilibrium that is reached is independent of the initial values \( q(0) \). In any event, the results with the initial values as given above are shown in Figure 3.

![Figure 3: The quantity of solute in each tank in the CODEE example.](image)

**The general case**

When the parameters are all nonnegative, it is easy to see that the trace of \( A \) is nonpositive and the det is nonnegative. With a bit more work it can also be shown that the discriminant of the quadratic characteristic polynomial is nonnegative so that both eigenvalues are real (and, from the det and trace information, are both nonpositive). Again, when \( \det(A) > 0 \), the equilibrium is approached at \( t \) tends to infinity.

Mathematica:
(* 2 tanks *)
ss[u_] := ExpToTrig[Expand[u]];

(* Fulford example *)
A = {{-4/100, 3/100}, {4/100, -5/100}};
Eigenvalues[A]
\%
Det[A]
expmAt = MatrixExp[t*A]
q0 = {{0}, {0}};
b1 = {{1}, {0}};
Ainv = Inverse[A];
qEquilib = Simplify[ -Ainv. b1]
tmp = qEquilib + expmAt . (q0 - qEquilib);
pfulford = Plot[{tmp[[1, 1]], tmp[[2, 1]]}, {t, 0, 240},
PlotStyle -> {RGBColor[1, 0, 0], RGBColor[0, 1, 0]}];

(* CodeE example *)
Acodee = {{-6/5, 1}, {3/5, -3/2}};
Eigenvalues[Acodee]
Det[Acodee]
expmAt = Exp[-27*t/20]*Map[ss, Map[Simplify,
Evaluate[Exp[27*t/20]*MatrixExp[t*Acodee]]]]
(* which gives a formula for expm(t A), codee example *)
q0 = {{3}, {5}};
b1 = {{1}, {1}};
Ainv = Inverse[Acodee];
qEquilib = Simplify[ -Ainv. b1]
tmp = qEquilib + expmAt . (q0 - qEquilib);
pcodee = Plot[{tmp[[1, 1]], tmp[[2, 1]]}, {t, 0, 8},
PlotStyle -> {RGBColor[1, 0, 0], RGBColor[0, 1, 0]},
AxesOrigin -> {0, qEquilib[[2, 1]]}];

(* general parameters *)
d1 = s1 - r12 + r21;
d2 = s2 + r12 - r21;
bVec = {{g1*s1}, {g2*s2}};
Agen2 = {{-(r12 + d1)/v1, r21/v2}, {r12/v1, -(r21 + d2)/v2}};
qEquilibgen2 = Map[Factor, Inverse[Agen2] . bVec];
Factor[Det[Agen2]]
(* test input argument below, but note lambda is fixed as indep variable *)
discrim[u_?(PolynomialQ[#, lambda] &)] := Resultant[u, D[u, lambda], lambda];
(* for monic poly in lambda *)
discrimgen2 = Collect[Expand[v1^2*v2^2*discrim[CharacteristicPolynomial[Agen2, lambda]]], {s1, s2}, Factor]
Collect[Expand[discrimgen2], {v1, v2}, Factor]
Collect[Expand[discrimgen2-(s1*v2-s2*v1)^2],{v1,v2},Factor]
Expand[-discrimgen2-((s1+r21)*v2-(s2+r12)*v1)^2 -4*v1*r12*v2*r21]
(* last line is 0, so that under the hypotheses that
all of v1,v2,s1,s2,r12,r21 are nonnegative
both eigenvalues are nonpositive. *)
2.2 General linear systems

In general the problem is to solve for $y(t)$ satisfying

$$\frac{dy}{dt} - A(t)y = f(t)$$  \hspace{1cm} (N)

with $y(0) = y_0$ given.

Here $A(t)$ is a given matrix-valued function of $t$, and $f(t)$, the ‘forcing’, is a given vector-valued function of $t$. Think of $t$ as ‘time’. This is called an initial-value problem.

**Definition.** When $f(t) \equiv 0 \forall t$,

$$\frac{dy}{dt} - A(t)y = 0$$  \hspace{1cm} (H)

we say (H) is a homogeneous equation, while (N) is called nonhomogeneous.

**MAJOR TOPICS**

1. A formula, the ‘variation of parameters’ formula, for solving (N) in terms of certain matrix solutions $\Phi(t)$ of

$$\frac{d\Phi}{dt} - A(t)\Phi = 0.$$

Nonsingular $\Phi$ satisfying this are called ‘fundamental matrices’. We treated this topic in the earlier section on matrix revision, matrix inverses.

2. The constant-coefficient case, the case

$$A(t) = A, \quad A \text{ a constant matrix.}$$

We will see similarity and eigenvalues used to determine its solution. In particular, the fundamental matrix which also satisfies $\Phi(0) = I$, the identity matrix, is denoted

$$\Phi(t) = \expm(tA).$$

3. Special methods, ‘method of undetermined coefficients’ for constant coefficients and special forms of the right-hand side $f$. Definitely not examinable in this unit, but if you ever have to do hand calculation, they are easier than variation of parameters.

**Definition.** A fundamental matrix ($\Psi$) for $\frac{dx}{dt} - A(t)x = 0$ is an $n$ by $n$ matrix whose columns are linearly independent solutions of the system.

Let $\Phi(t)$ be the matrix satisfying

$$\frac{d\Phi}{dt} - A(t)\Phi = 0, \quad \Phi(0) = I \text{ (identity matrix).}$$

This $\Phi$ is called the (principal) fundamental matrix (with initial time $t = 0$).

**CONSTANT COEFFICIENT EXAMPLES** Consider the two matrices $A$ and $M$, listed with their eigenvalues and eigenvectors below:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{e’val } \sqrt{2}, \text{ e’vec } \begin{bmatrix} 1 \\ 1 + \sqrt{2} \end{bmatrix}, \quad \text{e’val } - \sqrt{2}, \text{ e’vec } \begin{bmatrix} 1 \\ 1 - \sqrt{2} \end{bmatrix};$$

$$M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{e’val } 1, \text{ e’vec } \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{e’val } 0, \text{ e’vec } \begin{bmatrix} 1 \\ -1 \end{bmatrix}. $$
You can readily check that the following give fundamental matrices for each of the systems:

\[
\Psi_A = \begin{bmatrix}
\exp(t\sqrt{2}) & \exp(-t\sqrt{2}) \\
(1 + \sqrt{2})\exp(t\sqrt{2}) & (1 - \sqrt{2})\exp(-t\sqrt{2})
\end{bmatrix}
\]

\[
\Psi_M = \begin{bmatrix}
\exp(t) & 1 \\
\exp(t) & -1
\end{bmatrix}
\]

Neither of these is equal to the identity matrix at \( t = 0 \). So although they are fundamental matrices for their respective systems, they are not principal fundamental matrices (and, for constant coefficient systems we will call the principal fundamental matrix the matrix exponential, but more on this later).

### 2.3 Variation of parameters

One of the reasons the principal fundamental matrix is important is the variation of parameters formula: that

\[
y(t) = \Phi(t) y(0) + \int_0^t (\Phi(s))^{-1} f(s) ds,
\]

solves the initial-value problem (N) above.

This was mentioned before, where we gave it as an example of using matrix inverses. The variation of parameters formula (which reduces to the integrating factor method when \( n = 1 \)) requires us to know all the solutions of the homogeneous problem (H), but given that solves (after some integrations) the nonhomogeneous problem (N).

### 2.4 Constant coefficient homogeneous systems

An important class of differential equations for which the (principal) fundamental matrix, or equivalently the general solution, can be found is d.e.s with constant coefficients, i.e. \( A(t) \) is constant in \( t \). For constant matrices \( A \), the principal fundamental matrix is also called the exponential matrix of \( A \). The reason for this last name will become clear soon.

For the rest of the treatment of d.e.s here, \( A(t) \) is a constant matrix and we consider \( \frac{dy}{dt} = Ay \).

As in other areas, there are several ways through to solution using CA, relevant CA commands being:

<table>
<thead>
<tr>
<th>Student Matlab</th>
<th>Maple</th>
<th>Mathematica</th>
</tr>
</thead>
<tbody>
<tr>
<td>dsolve</td>
<td>dsolve</td>
<td>DSolve</td>
</tr>
<tr>
<td>expm(t*A)</td>
<td>exponential(A,t)</td>
<td>MatrixExp[ A t]</td>
</tr>
<tr>
<td>[V,D]=eig(A)</td>
<td>eigenvects(A)</td>
<td>Eigenvectors[ A ]</td>
</tr>
<tr>
<td>[J,P]=jordan(A)</td>
<td>jordan(A,’P’)</td>
<td>JordanDecomposition[ A ]</td>
</tr>
</tbody>
</table>

**EXAMPLE.** \( \frac{dy}{dt} = Ay \) with \( A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \)

The eigenvalues of \( A \) are \( +\sqrt{2} \) and \( -\sqrt{2} \). The principal fundamental matrix is

\[
\Phi(t) = \begin{bmatrix}
c - \frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & c + \frac{s}{\sqrt{2}}
\end{bmatrix}
\]

where \( s = \sinh(t\sqrt{2}) \) and \( c = \cosh(t\sqrt{2}) \).
Well, you can CHECK this (calculate $\Phi'(t) - A\Phi(t)$), but, at this stage, I haven't told you how to derive it. And, it is a bit messy, so return to the CAS given here later.

```plaintext
(* FILE /u/keady/WWW/Teaching/M2en/LecNotesLA/05EvalsGen/Mathematica/expmEx1mma.txt *)
A= {{-1,1},{1,1}};
{evals,evecsT}=Eigensystem[A]
evecs= {Transpose[{evecsT[[1]]}],Transpose[{evecsT[[2]]}]}
(* The evals are + or - Sqrt[2] 
The corresponding eigenvectors are given in evecs
The line below, which evaluates to the zero vector is a check *)
Map[Simplify, A.evecs[[1]] - evals[[1]]*evecs[[1]]]
```

```plaintext
expmtA=MatrixExp[t*A]
(* and, as it is a bit long, check against myExp 
s2=Sqrt[2];
   build c1 and c2 as vector solutions of system each having 0 in an appropriate place at t=0 
c1= Exp[evals[[1]]*t]*evecs[[1]]/evecs[[1]][[2,1]] - Exp[evals[[2]]*t]*evecs[[2]]/evecs[[2]][[2,1]]
Map[Simplify, D[c1, t] - A.c1] (* zero, checks *)
c2= Exp[evals[[1]]*t]*evecs[[1]]/evecs[[1]][[1,1]] - Exp[evals[[2]]*t]*evecs[[2]]/evecs[[2]][[1,1]]
c10= c1 /. {t->0}
c20= c2 /. {t->0}
(* make sure that we have identity matrix at t=0 *)
myExp = Transpose[{Flatten[c1]/c10[[1, 1]], Flatten[c2]/c20[[2, 1]]}]
Map[Simplify, D[myExp, t] - A.myExp]
checkIt=expmtA- myExp
ans= Simplify[checkIt]
(* ans = zero matrix, checks expmtA *)
```

We now return to general constant matrices $A$. In the case that $A$ is $1 \times 1$, this was solved in first year maths. $\frac{dy}{dt} = Ay$, $y(0) = 1$ has solution $\phi(t) = \exp(At)$ where exp can be defined by its taylor series

$$\exp(At) = 1 + \sum_{j=1}^{\infty} \frac{t^j}{j!} A^j.$$ 

This formula generalises to the case $A$ an $n \times n$ matrix, the $1$ being replaced by an identity matrix.

The (principal) fundamental matrix solving

$$\frac{d\Phi}{dt} = A\Phi , \quad \Phi(0) = I,$$

is, [StrBC] §5.4, p276, equation (3),

$$\Phi(t) = \expm(tA) = I + \sum_{j=1}^{\infty} \frac{t^j}{j!} A^j.$$
Though we won’t prove it, the series happens to be sufficiently rapidly convergent to allow the interchange of limit processes – differentiation and infinite summation. Believing this, and using the expm series for $\Phi$, we have

$$
\Phi'(t) = 0 + \sum_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!} A^j
$$

$$
= A \left( I + \sum_{j=2}^{\infty} \frac{t^{j-1}}{(j-1)!} A^{j-1} \right)
$$

$$
= A\Phi
$$

as required.

**EXAMPLES**. A $2 \times 2$ example

$$
A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \frac{\nu^j A^j}{j!} = \begin{bmatrix} \frac{\nu^j \lambda_1^j}{j!} & 0 \\ 0 & \frac{\nu^j \lambda_2^j}{j!} \end{bmatrix}, \quad \expm(tA) = \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix}.
$$

A larger example

$$
A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \expm(tA) = \begin{bmatrix} \exp(\lambda_1 t) & 0 & 0 \\ 0 & \exp(\lambda_2 t) & 0 \\ 0 & 0 & \exp(\lambda_3 t) \end{bmatrix}.
$$

We will learn easier ways – via matrix similarity and jordan forms – to evaluate $\expm(A,t)$. This is easier than the series when the jordan form is available. ([StrBC] 5L, p277 treats the case of the simplest jordan forms, i.e for $A$ diagonalizable.)

### 2.5 The relationship with eigenvalues

Consider

$$
y' = \frac{dy}{dt} = Ay.
$$

(H)

You are reminded that, if $u$ is an eigenvector of $A$, i.e $Au = \lambda u$, then a solution of (H) is given by

$$
y(t) = \exp(\lambda t)u.
$$

Easy Proof.

$$
\frac{d}{dt} \exp(\lambda t)u = \lambda \exp(\lambda t)u. \quad (1)
$$

and

$$
A \exp(\lambda t)u = \exp(\lambda t)Au = \lambda \exp(\lambda t)u. \quad (2)
$$

(1) and (2) together give the result.

If there is a full set ($n$) of linearly independent eigenvectors, we can start from these to construct a fundamental matrix, and thence the principal fundamental matrix $\expm(At)$. See the CAS – given previously – for the very first example of a principal fundamental matrix when we had
$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The general story if there are $n$ linearly independent eigenvectors $u_1, \ldots, u_n$ is that we construct a matrix solution $M$ of $\frac{dM}{dt} - AM = 0$ by

$$M(t) = \left[ \exp(\lambda_1 t)u_1 \mid \cdots \mid \exp(\lambda_n t)u_n \right],$$

after which

$$\expm(At) = M(t)M(0)^{-1}.$$ 

I prefer, though, to organize the calculations along the following lines – using similarity. (Actually, it is ‘imagining’ how the calculations are done: in practice, engineering mathematical software, e.g. Mathematica MatrixExp, etc., is used.) This ‘use similarity’ approach is general, not restricted to diagonalizable matrices $A$.

**Similarity, matrix exponentials, and jordan forms**

Strategy for solving $\Phi' = A\Phi$

1. Find a simpler $B$ similar to $A$, $B = P^{-1}AP$
2. Solve $\Psi' = B\Psi$
3. $\Phi = P\Psi$ or, since $\Phi C$ is solves the same d.e. as $\Phi$ does, $\Phi = P\Psi P^{-1}$ will also work as a solution.

Check 3 works. (The steps work equally well without the final $P^{-1}$.)

$$\Phi' = P\Psi P^{-1} = PB\Psi P^{-1} \quad \text{using 2,}$$

$$= P P^{-1} AP \Psi P^{-1} \quad \text{using 1,}$$

$$= A \Phi \quad \text{using definition 3 of } \Phi,$$

as required.

**SUMMARY:** If $P^{-1}AP = B$, we have

$$A = PB P^{-1} \quad \text{and } \expm(tA) = P \expm(tB) P^{-1}.$$ 

**EXAMPLES** $\Psi = \expm(tB)$ for simple $B$.

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \\ \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \expm(tB) = \begin{bmatrix} \exp(\lambda t) & 0 \\ 0 & \exp(\mu t) \\ \exp(\lambda t) & \begin{bmatrix} 1 \\ t \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad \expm(\alpha t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

It turns out that this is essentially all the different sorts of behaviour that are possible. Indeed the last matrix is similar to

$$D = \begin{bmatrix} \alpha + i\omega & 0 \\ 0 & \alpha - i\omega \end{bmatrix}, \quad \expm(Dt) = \exp(\alpha t) \begin{bmatrix} \exp(i\omega t) & 0 \\ 0 & \exp(-i\omega t) \end{bmatrix},$$

so that there are really only two different sorts of solution behaviour in $\Phi(t)$ exponentials in $t$ and (polynomials in $t$) times (exponentials in $t$).
Numerical EXAMPLE. Find $\expm(tA)$ when $A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$.

Solution. First, as Mathematica JordanDecomposition, will readily tell you, we have

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = PBP^{-1}.$$ 

Following the strategy above we have

$$\expm(tA) = P \begin{bmatrix} \exp(3t) & 0 \\ 0 & \exp(5t) \end{bmatrix} P^{-1} = \begin{bmatrix} \frac{3}{2} \exp(3t) - \frac{1}{2} \exp(5t) & -\frac{1}{2} \exp(3t) + \frac{1}{2} \exp(5t) \\ \frac{3}{2} \exp(3t) - \frac{1}{2} \exp(5t) & -\frac{1}{2} \exp(3t) + \frac{1}{2} \exp(5t) \end{bmatrix}.$$ 

EXERCISE. Check the details in the preceding example.

Another Numerical EXAMPLE. For the matrix $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

whose eigenvalues and eigenvectors, and for which a fundamental matrix was given before, find $\expm(Mt)$ beginning with

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = P^{-1} \quad \text{and} \quad P^{-1}MP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = B.$$

Solution.

$$\expm(tM) = P \expm(tB) P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \exp(t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \exp(t) + 1 & \exp(t) - 1 \\ \exp(t) - 1 & \exp(t) + 1 \end{bmatrix}.$$ 

Carrying out the above strategy in general takes us back to the question we asked after looking at diagonalization.

What simple form can any $n \times n$ matrix be reduced to by similarity?

I.e. given $A$, find a simple form for $B = P^{-1}AP$? The answer, from before, is as follows.

JORDAN FORM THEOREM. [StrBC] 5U, p312. An $n \times n$ matrix $A$ with eigenvalues, repeated with their algebraic multiplicities, $\lambda_1$, $\lambda_2$, ..., $\lambda_n$ is similar to a matrix with these eigenvalues in the diagonal positions, zeros and ones along the superdiagonal and zeros elsewhere.

Remark. The number and location of the 1s depends on the difference between algebraic and geometric multiplicities.

The easiest nondiagonalizable EXAMPLE. Let $Z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Find $\expm(tZ)$.
Solution. The easiest way is, in the series for \( \expm(Zt) \), to use the fact that \( Z^2 = 0 \) and hence \( Z^j = 0 \) for all \( j \geq 2 \). Thus the series has only 2 terms

\[
\expm(tZ) = I + tZ = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.
\]

This easy example with \( A \) not diagonalizable is given in [StrBC] Q5.4.7, p287.

The general form of \( \expm(Jt) \) for a Jordan block \( J \) is given in [StrBC] §5.6, p314.

A Harder Numerical EXAMPLE. For the defective matrix we have looked at before

\[
A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}
\]

find \( \expm(tA) \).

Solution, via Mathematica.

\[
A= \{(1,-1,0),(0,1,1),(0,0,-2)\};
\]

\[
\text{expmAt= MatrixExp[t*A]}
\]

\[
\{P,J\}=\text{JordanDecomposition}[A];
\]

\[
(* \text{From before, the jordan form is} \)
\]

\[
J = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\text{Pinv=Inverse[P];}
\]

\[
\text{expmJt= MatrixExp[t*J];}
\]

\[
\text{checkIt=Simplify[expmAt-P.expmJt.Pinv]}
\]

Cutting and pasting from \LaTeX{} produced from the above:

\[
\expm(tA) = \begin{bmatrix} e^t & -te^t & -1/3te^t + 1/9 e^t - 1/9 e^{-2t} \\ 0 & e^t & 1/3e^t - 1/3 e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}
\]

We remark that, as \( A \) is upper triangular, the series shows that \( \expm(At) \) will also be upper triangular.

The behaviour with the polynomials in \( t \) multiplied by the exponentials is typical of what happens when \( A \) is defective (i.e. not diagonalable).

Observe that the variation of parameters formula for constant \( A \) does not actually involve any hard calculations of matrix inverses as, if

\[
\Phi(t) = \expm(tA), \quad \text{then } \Phi(s)^{-1} = \Phi(-s).
\]

Various other comments on matrix exponentials are possible

\[
AB = BA \quad \implies \quad \expm(A + B) = \expm(A)\expm(B).
\]

(See also [StrBC] §5.4, Q5.4.5 p287 and 5F on commuting matrices and simultaneous diagonalizability, or see [StrWC] p261 Thm 6F concerning commuting matrices sharing eigenvectors.)
2.6 Method of undetermined coefficients

This is a generalisation to constant coefficient systems of the method with the same name you learnt for a single d.e. in first year. It is, when applicable, MUCH easier than the more general method of variation of parameters. (You may also use Laplace transforms for these problems. See your M132 work.)

Let’s begin with an example. Find a particular solution of
\[
\frac{dy}{dt} - Ay = \exp(\alpha t)r,
\]
when \(\alpha\) is NOT an eigenvalue of \(A\).

Look for solution \(y(t) = \exp(\alpha t)s\). Substituting, we have
\[
\alpha s - As = r, \quad \text{or} \quad (\alpha I - A)s = r,
\]
and this has a unique solution since \(\alpha I - A\) is invertible.

Remark. The case when the rhs is a constant vector \(r\) is the special case \(\alpha = 0\).

This generalises.

Conclusions. The algorithm for solving constant coefficient linear systems of d.e.s
\[
\frac{dy}{dt} - Ay = f,
\]
is precisely defined and, being just routine, calculation can be implemented in computer algebra programs.

Although in the past engineers had to be able to do hand calculations for problems like that on previous page, you can now leave tedious details to mathematical software.

An understanding will always be necessary. In particular both hand-calculation and computer algebra software depend on being able to find eigenvalues. If these can only be found approximately numerically you may have to work harder. The shape of the final solution will also depend on whether exact expressions for eigenvalues are short or not.
2.7 $y' = Ay$: Eigenanalysis of the stability of the zero solution

As is perhaps correct in a first account, our treatment of $\expm(tA)$, so far, has concentrated on actually calculating it. However, to do this exactly, one has to find the eigenvalues exactly, and this is often not possible. Fortunately there is easily extracted useful information without having the eigenvalues exactly calculated.

**Definition.** A square matrix $A$ is said to be *stable* if the real parts of all its eigenvalues are negative.

The zero solution of $y' = Ay$ is an ‘equilibrium point’. A question is: ‘is it stable?’

It is worth beginning with the $n = 1$ case: $A$ is just a real number. If $A < 0$, the solution $\exp(tA)y(0) \to 0$ as $t \to \infty$ so the zero solution is stable. I.e. start near it, and the system takes you closer to it. If $A > 0$, the zero solution is unstable. Let’s now return to systems. The general result is that the zero equilibrium (solution) is stable iff the matrix $A$ is stable.

There is quite a lot more that can be said. Here we confine our further remarks to the case where $A$ is 2 by 2. (Of course, now we could actually find the eigenvalues exactly if we wished to do so.) Perhaps you might regard this ‘more that can be said’ as telling you a bit more about how the solutions behave close to the equilibrium. To give a 1st year example, the damped linear oscillator has 0 as a stable solution, but the solutions look different in the underdamped and the overdamped cases.

### 2.7.1 A couple of examples of stability for $\dot{y} = Ay$, $A$ 2 by 2

**EXAMPLE.**

$$A = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix}, \quad \expm(tA) = \exp(\alpha t) \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

$A$ is a stable matrix iff $\alpha < 0$.

What do the solutions look like as $t$ increases?

Describe the trajectory $(y_1(t), y_2(t))$ in the $(y_1, y_2)$-plane if we start from $(1, 0)$.

**Solution.** Since $\alpha < 0$, any solution tends to zero as $t \to +\infty$.

For the solution starting at $(1, 0)$ we have

$$y(t) = \exp(\alpha t) \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}$$

So, setting $r^2 = y_1^2 + y_2^2$, we have

$$r^2 = \exp(2\alpha t), \quad \theta = \arctan \left( \frac{y_2}{y_1} \right) = \omega t$$

In polar coordinates this is

$$r = \exp(\alpha \theta / \omega)$$

which, with $\alpha < 0$ is a spiral tending to zero as $\theta$ increases.

**EXAMPLE.** Consider the differential equation, for an unforced damped linear oscillator:

$$y'' + \frac{1}{2}y' + \frac{257}{16}y = 0$$
or, equivalently, with \([y_1; y_2] = y = [y; y']\),

\[
y' = Ay \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ \frac{-257}{16} & -\frac{1}{2} \end{bmatrix}.
\]

Using Mathematica, one finds that the eigenvalues of \(A\) are \(-\frac{1}{4} \pm 4i\), and that

\[
\expm(tA) = \exp(-t/4) \begin{bmatrix} \cos(4t) + \frac{1}{16} \sin(4t) & \frac{1}{2} \sin(4t) \\ \frac{-257}{64} \sin(4t) & \cos(4t) - \frac{1}{16} \sin(4t) \end{bmatrix}.
\]

Now suppose that the system is started with \(y(0) = [0; 1]\), i.e. \(y(0) = 0, y'(0) = 1\).

- Write down the formula for \(y(t)\) solving this initial value problem.
- How many times does \(y(t) = y_1(t)\) pass through 0 when \(t\) varies from 0 up to 5?
- Sketch, with attention to the main qualitative features, but without too much concern about detailed numerics, the general shape of the trajectory of the solution, of the first part above, in the \((y_1, y_2)\)-plane.

Mathematica’s ParametricPlot or similar is the easiest way to get to the drawing of the spiral, but, if you don’t have convenient access to this at present, you may use the following table to get a few points on the plot:

<table>
<thead>
<tr>
<th>(t)</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y_1(t))</td>
<td>0.00</td>
<td>0.19</td>
<td>-0.08</td>
<td>-0.07</td>
<td>0.06</td>
<td>0.06</td>
<td>0.05</td>
<td>0.03</td>
<td>-0.02</td>
<td>0.06</td>
</tr>
<tr>
<td>(y_2(t))</td>
<td>1.00</td>
<td>-0.43</td>
<td>-0.47</td>
<td>0.67</td>
<td>-0.13</td>
<td>-0.43</td>
<td>-0.41</td>
<td>0.03</td>
<td>-0.35</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Solution. With \(\expm(tA)\) as given above

\[
y(t) = \expm(tA) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

so

\[
y_1(t) = \exp(-\frac{t}{4}) \sin(4t).
\]

From this we can find the number of zeros for the interval of \(t\), \(0 < t \leq 5\). It is the same as the numbers of zeros of \(\sin(x)\) (where \(x = 4t\)) with \(0 < x < 20\). Since \(6\pi \approx 18.85\) and \(7\pi \approx 22\), there are, after starting at zero, 6 more zeros of \(y_1\) before \(t = 5\).

To do the plot the easy way, using Mathematica:

```mathematica
y1 = Exp[-t/4] Sin[4*t]/4;
y2 = D[y1, t];
ParametricPlot[{y1, y2}, {t, 0, 5}, AspectRatio -> Automatic];
```

The result of running the code is the spiral shown in Figure 4.

The examples in this subsection have both sketches/plots as spirals. Other sorts of behaviour are possible. It isn’t hard – merely a bit long – to check out all the various possibilities for 2 by 2 matrices. The next subsection will have an account of this.
Figure 4: Example of a phase portrait for a underdamped linear pendulum. Matlab has labelled the axes $x$ corresponding to our $y_1$ and $y$ corresponding to our $y_2$. 

2.7.2 General comments about $\dot{y} = Ay$, $A$ 2 by 2

**THEOREM.** Let $A$ be a 2 by 2 real matrix, $p$ is the trace of $A$ and $q = \det(A)$. Let $D = p^2 - 4q$. There are three possibilities for the eigenvalues of $A$.

- The eigenvalues of $A$ are real and distinct ($D > 0$).
- The eigenvalues of $A$ are real and equal ($D = 0$).
- The eigenvalues of $A$ are a complex conjugate pair ($D < 0$).

(In the damped nonlinear pendulum example, given later in these notes, the first case corresponds to the unstable upwards equilibrium, the last to the downwards equilibrium.)

The proof of the theorem follows from calculating the eigenvalues of $A$. They are

\[
\lambda_{\pm} = \frac{1}{2} \left( \text{trace}(A) \pm \sqrt{\text{trace}(A)^2 - 4\det(A)} \right)
= \frac{1}{2} \left( p \pm \sqrt{p^2 - 4q} \right).
\]

There are only a few different sorts of behaviours in the phase-plane for 2nd-order constant coefficient systems. See Figure 5. The different cases can be listed.

- $\det(A) < 0$. The equilibrium point is a **saddle**.
- $\det(A) > 0$.
  - $\text{trace}(A) > 0$. The equilibrium point is a **source** (unstable).
  - $\text{trace}(A) < 0$. The equilibrium point is a **sink** (stable).
  - $D < 0$. The equilibrium point is a **spiral**.
  - $D > 0$. The equilibrium point is a **node**.
  - $D = 0$. $A = cI$, the equilibrium point is a **focus**.
    - $D = 0$. $A \neq cI$, the equilibrium point is an **improper node**.
- $\det(A) > 0$ and $\text{trace}(A) = 0$. **centre**: nonzero purely imaginary eigenvalues.

- $\det(A) = 0$. Three subcases:
  - single zero eigenvalue: **saddle-node**;
  - double zero eigenvalue, one-dimensional nullspace: **shear**;
  - $A = $ zero matrix.

References


Figure 5: The different sorts of equilibrium points. StrBC Fig 5.2 p281, O’Neil AEM Fig 11.47.