Intro Lec 01

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(o) d.e.s

\[ y' = Ay \]
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Topics Weeks 4-6

- d.e.s
- determinants, eigenvalues revisited
  - matrix exponentials
- 2 d examples
  - NDSolve, phase portraits, and
  - how linear theory helps with understanding nonlinear d.e.
Topics Weeks 8

- orthogonality,
- eigenproperties of real symmetric matrices
- normal modes of oscillation
Topics Weeks 11-13

- matrix exponentials again
- stability of equilibria for some nonlinear systems
- orthogonality in function spaces
  Fourier Series (probably briefly),
  small amplitude nonlinear oscillations
Interested in linear d.e.s and how they can help us learn about nonlinear ones. All are easy enough to approximate numerically. Mathematica `NDSolve`. Easiest example (linear):

\[ y' = Ay \]

is solved by \[ y(t) = \exp(At)y(0) \]

- \( A > 0 \) solution gets bigger and bigger in magnitude as \( t \) increases.
- \( A < 0 \) solution gets smaller and smaller in magnitude as \( t \) increases.
First order d.e. examples

Let’s start with examples of a single first order d.e.
In MATH1020 you treated:

- direction fields for these,
- exact solution for
  - separable d.e.s
  - linear d.e.s

\[ y' = Ay \]
A nonlinear example

Euler’s disk. Stopping in finite time.
The simple d.e. describing the angle $\alpha$ to the horizontal is

$$\dot{\alpha} = -\frac{\epsilon}{\alpha^n}$$

with $\epsilon$ a positive quantity. The solution is

$$\alpha(t_0)^{n+1} - \alpha(t)^{n+1} = (n+1)\epsilon(t - t_0)$$

The solution reaches zero at

$$t_1 = t_0 + \frac{\alpha(t_0)^{n+1}}{(n + 1)\epsilon}.$$ 

In the case $n = 2$, the finite-time stopping is shown in Figure 1.
The Euler disk ‘toy’

Will be able to see/hear the sudden stopping with the actual toy in the lab.
(In principle, I could code Mathematica to make the right sound ...)

The previous d.e. associated with a rough and ready model of the dissipation is far from the only modelling that has been done.
The Euler disk ‘toy’, week 11

We will look at nondissipative motions later ... about week 11 or 12
4th order nonlinear system ...
Possibly relevant to some details
I. of the dynamics
II. of sound generated.
Problems get less difficult if one looks at them in appropriate ways and doesn’t try to solve everything.
Rolling disk, energy

Energy considerations are relevant both for the dissipative d.e. before and our later mechanics d.e. sums. In these later sums, I will derive formulae for the kinetic energy $T$ and potential energy $V$. You will get some practice at finding $T$ and $V$ in various other problems. The total energy $E = T + V$. The derivation of the preceding $\alpha(t)$ 1st order nonlinear d.e. for the Euler disk starts from an expression for $E(\alpha)$ for very simple motions. More controversial is the model for the dissipation. Moffatt (2000) *Nature* 204 p833.
Figure: Plot of $\alpha(t) = (1 - t)^{1/3}$. $\varepsilon = 1/3$, $t_0 = 0$, $\alpha(0) = 1$. 
d.e.s of various orders

The lab this year will be revision of 1st year on mass-spring oscillator sums.
2nd order.
linear
It is too nice and too important a problem for me to ignore. I hope I will be saying things you have already met. It is good to revisit maths you have already seen with mathematica around to draw pictures. Hopefully you will be re-assured by it. Of course, we the lecturers can’t do too much of this as you need to learn new stuff too. Some more introductory items before the ‘linear oscillator’…
Recognizing linear d.e.s

<table>
<thead>
<tr>
<th>Notation here: $y'$ means $\frac{dy}{dt}$</th>
<th>de</th>
<th>order</th>
<th>linear?</th>
<th>if linear homogeneous?</th>
<th>if linear const.coefft?</th>
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<tbody>
<tr>
<td>$y' - y = 0$</td>
<td>1</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
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<tr>
<td>$y' - y = 1$</td>
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<td>$y' - ty = 0$</td>
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<td>Y</td>
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<tr>
<td>$y' - y(1 - y) = 0$</td>
<td>1</td>
<td>N</td>
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<tr>
<td>$my'' + cy' + ky = 0$</td>
<td>2</td>
<td>Y</td>
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<tr>
<td>$my'' + cy' + ky = f(t)$</td>
<td>2</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
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</tr>
</tbody>
</table>
Chpt 1, 1st order
Treated in 1st year
  ▶ linear (integrating factor method = variation of parameters)
  ▶ separable
  ▶ in general, can’t find formulae for soln
    NDSolve numeric soln always available.
    Basic idea for numerics visible in ‘direction fields

See the handout notes.
Read the handout. Much of it should be revision.
Do the Lab. (Will cover material useful in physics and applied maths).
And, illustrating that it isn’t easy to teach maths in a logical one-thing-after-another order, listen in lectures as I push the line ‘vector spaces are good for understanding d.e.s’.
(And it is easiest to show this with linear d.e.s first.)
There is ‘big picture’ aspects – vector spaces, linearity.
There are also little details, what solutions to particular problems are really like, formulae, plots, etc., etc..
In physics and applied maths both matter.
As a lecturer, my guess is that my main job is to try to get the ‘big picture’ ideas over.
Vector spaces, operators on them - crucial for quantum mech, etc.
An easy numerical example, (H)

(This is given a bit later in printed notes.)

\[ y'' + 2y' + 5y = 0, \quad (H) \]

a homogeneous d.e., has real solutions

\[ y_h(t) = C_1 \exp(-t) \cos(2t) + C_2 \exp(-t) \sin(2t) \]

for any real constants \( C_1, C_2 \).

In other words, the solutions of (H) form a (function) vector space whose basis is \( \{ \exp(-t) \cos(2t), \exp(-t) \sin(2t) \} \).

We will soon give (repeat!) the definition of ‘vector space’.
Mathematica solving (H)

Routine calculation is automated, e.g. in Mathematica:

\[ yh\text{Sol} = \text{DSolve}\left[ D[y[t],\{t,2\}] + 2*D[y[t],t] + 5*y[t] == 0, y[t], t \right] \]
\[ yh = \text{Factor}\left[ y[t] \/. yh\text{Sol}[[1]] \right] \]; \text{InputForm}[yh]

\[
(* yh = \\
(C[2]*\cos[2*t] + C[1]*\sin[2*t])*\exp[-t]
*)
\]

Do the plots of the solution in Mathematica (if time).
Solve i.v. problems.
Figure: Solutions of $y'' + cy' + y = 0$. $c = 0$ would give a sinusoidal solution. $c > 0$ small (underdamped) shown.
The lab has you doing the plots

Different situations.

- lightly damped (see oscillations, previous example)
- critically damped (the damping is such that if it were to be any less you would have oscillations)
- overdamped: no oscillations. Roots of the auxilliary equation distinct and real.

Lots of applications. E.g. restaurant swing door - choose mechanical parameters so that it is somewhere around critically damped ...
Overdamped, etc.

Figure: Solutions of $y'' + cy' + y = 0$. $c = 2$, critical damping

? Restaurant swing doors
Overdamped, etc.

Figure: Solutions of $y'' + cy' + y = 0$. $c > 0$ large (overdamped)
An easy numerical example, (N)

\[ y'' + 2y' + 5y = F \cos(\omega t) = F \text{Re} \left( \exp(i\omega t) \right), \quad (N) \]

a nonhomogeneous d.e., has real solutions \( y(t) = y_h(t) + y_p(t) \) where

\[ y_p(t) = F \text{Re} \left( \frac{\exp(i\omega t)}{5 + 2i\omega - \omega^2} \right) \]

is a *particular solution* and \( y_h \) is the general solution of the homogeneous problem. Re means ‘real part’.

In other words, there are similarities with the situation with linear *algebraic* equations.
Digression on details - Lab

\[ y'' + cy' + y = F \cos(\omega t) = F \text{Re} (\exp(i\omega t)) \], \quad (N)

Will just talk about what you will see in the lab plots. Actually just a digression from my main point on linearity coming soon.
Figure: Solutions of $y'' + cy' + y = \cos(wt)$. $c = 0$, $w = 1$ is resonance
Digression - Lab - Damped forced

Figure: Solutions of $y'' + cy' + y = \cos(\omega t)$. $w = 1$. $c > 0$
Figure: Solutions of $y'' + cy' + y = \cos(w t)$. $c = 0$, $w = 15/16$, beats.

Piano tuner?
A nonhomogeneous d.e. has solutions \( y(t) = y_h(t) + y_p(t) \) where \( y_p \) is a particular solution and \( y_h \) is the general solution of the homogeneous problem. In other words, there are similarities with the situation with linear algebraic equations.
Review of basic linear algebra

- With $A$ a $m \times n$ matrix of real numbers, $\mathbf{0}$ a $m \times 1$ (column) vector of zeros, the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is a vector space (actually a subspace of $\mathbb{R}^n$). (In first year, you learnt that this subspace is called the *nullspace* of $A$.)

- If you know one solution $\mathbf{x}_p$ of the nonhomogeneous equation $A\mathbf{x}_p = \mathbf{b}$, then any other solution is of the form $\mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the homogeneous equation $A\mathbf{x}_h = \mathbf{0}$.

- The equation $A\mathbf{x} = \mathbf{b}$ has either no solution, precisely one solution or infinitely many solutions.
Definition of ‘vector space’, 1

Let $\mathbb{K}$ is a field (the rationals $\mathbb{Q}$, the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$ for us). The elements of $\mathbb{K}$ will sometimes be called *scalars*.

**Definition.** Let $V$ be a set. We say that $V$ is a *vector space over* $\mathbb{K}$ if (a) and (b) below are satisfied.
(a) Addition and multiplication by a scalar are defined on $V$, i.e.

$$x + y \in V, \quad \lambda x \in V, \quad \forall x, y \in V, \lambda \in \mathbb{K}.$$

(b) The following 10 postulates ($A0$), . . . , ($M4$) are satisfied:
Definition of ‘vector space’, A

(A0) \( V \) is closed under addition, \( \forall x, y \in V, x + y \in V \);
(A1) addition is associative on \( V \), \( \forall x, y, z \in V, (x + y) + z = x + (y + z) \);
(A2) addition is commutative on \( V \), \( \forall x, y \in V, x + y = y + x \);
(A3) there is a \( 0 \in V \) satisfying \( x + 0 = x \ \forall x \in V \);
(A4) negatives exist, \( \forall x \in V \exists y \in V \) such that \( x + y = 0 \);
Definition of ‘vector space’, M

(M0) $V$ is closed under scalar multiplication, $\forall x \in V, \forall \lambda \in \mathbb{K}$, $\lambda x \in V$;
(M1) $\forall x, y \in V, \forall \lambda \in \mathbb{K}$, $\lambda (x + y) = (\lambda x) + (\lambda y)$;
(M2) $\forall x \in V, \forall \lambda, \mu \in \mathbb{K}$, $(\lambda + \mu)x = (\lambda x) + (\mu x)$;
(M3) $\forall x \in V, \forall \lambda, \mu \in \mathbb{K}$, $\lambda (\mu x) = (\lambda \mu)x$;
(M4) $\forall x \in V$, $1x = x$. 

Grant Keady
MATH2200 Applied Maths Intro, ODEs and Linear Algebra
\( \mathbb{R}^n \) example

\[
\mathbb{R}^n = \text{set of } n - \text{tuples of real numbers}, \\
= \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}.
\]

With obvious definitions of addition and of multiplication by a real number \( \mathbb{R}^n \) is a (real) vector space.

Perhaps the simplest instance of this is the case \( n = 2 \).

\[
\mathbb{R}^2 = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}
\]

The definition of addition is
\[(x_1, x_2) + (X_1, X_2) := (x_1 + X_1, x_2 + X_2)\]
and the definition of scalar multiplication is
\[\alpha(x_1, x_2) := (\alpha x_1, \alpha x_2).\]
Continuous functions, function space example

Consider continuous real-valued functions defined on a real interval \([a, b]\). Denote the set of all of these by \(C([a, b])\). For \(f, g\) in \(C([a, b])\), and \(\lambda\) in \(\mathbb{R}\), define \(f + g\) and \(\lambda f\) by

\[
(f + g)(x) = f(x) + g(x),
\]

\[
(\lambda f)(x) = \lambda f(x).
\]

Then \(C([a, b])\) is a vector space.
Other vector spaces of functions, etc.

See printed notes

$C^k([a, b])$

$UP(x, n)$

$TP(x, n)$

Spaces of sequences ...

All these have their uses, e.g. Fourier Series
Definition of ‘subspace’

Let $V$ be a vector space over a field $\mathbb{F}$. A subset $W$ of $V$ which is also a vector space over $\mathbb{F}$ is called a *subspace* of $V$.

An equivalent characterisation is given in StrBC §2.1 p64. A set $S$ of vectors of $V$ is a subspace of $V$ if and only if

1. $0 \in S$;
2. (i) the sum of any two vectors in $S$ is in $S$, and (ii) the product of any scalar with any vector in $S$ is in $S$. (Actually (1) is a consequence of (2)(ii) with 0 as the scalar.)
Example

Let $S$ be a subspace of $\mathbb{R}^2$. Then there are three possibilities:

0. $S$ is the trivial subspace containing only the zero vector;
1. there is a straight line $L$ through the origin such that $S$ consists exactly of the vectors parallel to $L$.
2. $S = \mathbb{R}^2$. 
Example, ctd

Figure: Subspaces of $\mathbb{R}^2$
1st year LA extends to vector spaces in general. In particular, one can define

- *linear combination of vectors*
- \( \text{span}(S) \), the *span* of a set \( S \) of vectors
- *linearly dependent*, *linearly independent*, *basis*
- *dimension*
Definition. A vector space with a finite basis is said to be *finite dimensional*; otherwise it is *infinite dimensional*.

After giving the definition of vector space we saw two examples: $\mathbb{R}^n$ which is finite dimensional, and the function space $C([a, b])$ which is infinite dimensional.
## More that is in the Unit

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<th>generalising</th>
<th>1st course in LA</th>
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<td>(Abstract) vector spaces</td>
<td>generalise</td>
<td>$\mathbb{R}^n$</td>
</tr>
<tr>
<td>Normed vector spaces</td>
<td>generalise</td>
<td>$\mathbb{R}^n$ with distances</td>
</tr>
<tr>
<td>Inner-product vector spaces</td>
<td>generalise</td>
<td>$\mathbb{R}^n$ with dot products</td>
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</table>

You learnt in your 1st course in LA that dot products give you a method of treating angles between vectors. Inner product function spaces is the setting for Fourier Series, a topic treated later in this unit.
First results

\[(Ly)(t) := y'' + P_1(t)y' + P_0(t)y.\]

Defined in first year, linear 2nd order d.e.:

\[(Ly)(t) = f(t) \quad (N)\]

\[(Ly)(t) = 0 \quad (H)\]

(H) is a \textit{homogeneous} d.e.: (N) stands for \textit{non-homogeneous}.

**THEOREM.** The set of all solutions of a homogeneous linear d.e. forms a vector space.
THEOREM. Suppose that we have a particular solution $y_p$ of $Ly_p = f$. Then any (other) solution of $Ly = f$ is the sum of $y_p$ and a solution $y_h$ of the homogeneous d.e. $Ly = 0$.

These results are exactly like the corresponding results for algebraic equations given earlier. We stated them for 2nd order d.e.s, but they will work just as well for $n$-th order d.e.s for other values of $n$, including for $n = 1$. 
A result which is much more difficult to prove (so proof is definitely not for this unit) is:

**THEOREM.** The space of all solutions of a homogeneous linear $n$th-order d.e. is $n$ dimensional.
The steps in solving linear d.e.s

- Find all solutions – the solution space – for the homogeneous d.e..
- Find a particular solution of the nonhomogeneous d.e..
- Put the preceding bits together.
Advert: Variation of parameters

Once you know all the solutions of the corresponding homogeneous d.e., the ‘variation of parameters formula’ (which we will treat later in the unit) solves nonhomogeneous d.e.s.

This generalises what you did, in first year, for a single first order d.e.. Here is a reminder. Consider

\[
\frac{dy}{dt} + P(t)y = b(t). \tag{N}
\]

Suppose \( \phi \) is a nonzero solution of

\[
\frac{d\phi}{dt}(t) + P(t)\phi(t) = 0 \quad \text{for all } t. \quad \phi(t_0) = 1 \quad \tag{H}
\]

Then the solution of (N) with \( y(t_0) = y_0 \) is given by

\[
y(t) = \phi(t) \left( y_0 + \int_{t_0}^{t} \phi(s)^{-1} b(s) ds \right)
\]
This 1st order formula generalises! For now, it suffices to record:

SLOGAN. If you can find all the solutions of the homogeneous problem, there is a formula (‘variation of parameters’) for writing the solution of the nonhomogeneous problem in terms of integrals.

(CAUTION. Don’t do variation of parameters by hand. If you must do hand calculation, the ‘method of undetermined coefficients’ which you were shown in first year, when it is applicable, is easier. However, we recommend you use Mathematica.)
Solving the homogeneous d.e.?

How can we find formulae to solve the homogeneous d.e.? For d.e.s of higher order than first, in fact, the answer is ‘you usual cannot’. However, there is an important special case (and the only one treated in this unit), the case of constant coefficients.

SLOGAN. Homogeneous constant coefficient d.e.s have (possibly complex) exponential solutions.
Solving const. coefft homogeneous d.e.

EXAMPLE
Repeated for the umpteenth time!
(A simple special case of what is in the notes)

\[ my'' + ky = 0 \]

Look for solutions \( y(t) = \exp(rt) \).

\[ mr^2 + k = 0 \]

Thus solutions are

\[ y(t) = C_1 \exp(i \sqrt{\frac{k}{m}} t) + C_2 \exp(i \sqrt{\frac{k}{m}} t) \]

and the trivial matter of getting to real solutions was treated in first year.
Practical applications of 2nd order const coefft d.e.

See your earlier lectures, covering resonance and so on.
See also the printed notes.
See also Lab 4.
It was treated in 1st year. Revise it.
More.. superposition, etc.

Superposition is the key ingredient in determining the periodic response to periodic forcing for a (stable) constant coefficient (system of) d.e. From the Fourier Series of the input (forcing) it is easy to find the Fourier Series of the output (response).