Some conservative mechanical systems of the form
\[ \dot{\theta} = M_1 \omega, \quad \dot{\omega} = L \omega \times M_2 \omega - g(\theta) : \]
Equilibria and nonlinear oscillations

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1 Introduction

The mechanical systems treated are described by differential equations of the form
\[ \dot{\theta} = M_1 \omega, \quad \dot{\omega} = L \omega \times M_2 \omega - g(\theta). \] (1.1) \{eq:gen\}

Here \( \theta \) and \( \omega \) are 3-vectors, \( M_1 \) and \( M_2 \) are invertible diagonal matrices and \( L \) is a 3 by 3 matrix. The coefficients in \( M_1 \), \( M_2 \) and \( L \) may depend on \( \theta \).

The \( g(\theta) \) represents gravity. Our treatment of these mechanical systems is, or at least will be, in a series of notes structured as follows.

- A bead on a rotating hoop
  (not in the 2007 handouts, but in .nb format for labs).
- A disk rolling on a horizontal plane,
  I. General.
- The spherical pendulum
  (not in handouts, possibly for 2008 labs).
  A sphere rolling inside a sphere.
- A disk rolling on a horizontal plane,
  II. Motions with the disk nearly flat.
- A disk rolling inside a spherical shell.

1.1 Equilibria and stability

The general setting concerns, with the function \( f \) given,
\[ \frac{d\mathbf{y}}{dt} = f(\mathbf{y}). \]
We define *equilibrium points* as vectors $y_e$ where $f(y_e) = 0$. We consider stability by setting up a problem for the remainder $\rho$ where $y = y_e + \rho$, which, on linearizing gives

$$\frac{d\rho}{dt} = J\rho \text{ where } J = Df(y_e),$$

and $J$ is the Jacobian of $f$ evaluated at $y_e$.

One classifies the stability of an equilibrium, of course, by determining the signs of the real parts of the eigenvalues of $J$. If all the real parts are negative, the equilibrium is stable: if all the real parts are nonpositive, the equilibrium is neutrally stable. If any real part is positive, the equilibrium is unstable.

The steps proceed as follows.

- finding the equilibria, (a nonlinear algebraic system to solve),
- calculate the Jacobian and evaluate at the equilibria, and proceed to determine their stability. (The stability calculations should be viewed as elementary linear algebraic ones.)

### 1.2 Numerical solution

Nonlinear d.e.s may have intricate global behaviour. The equilibria, and local analysis near them, give important information. Also straightforward is the numerical solution of the full equations. The downside of numerical solution is that one can have too many numbers and plots, and a qualitative framework of what the solutions are like is also needed.

### 1.3 Integrable systems, special function solutions, and asymptotics

The bead on a rotating hoop, the disk rolling on a horizontal plane, the spherical pendulum and the sphere rolling inside a sphere are all integrable systems, and this observation can be used to check the numerics.

In all of the problems, the nonlinear pendulum equation,

$$\ddot{\theta} + \left(\frac{g}{l}\right) \sin(\theta) = 0$$

occurs as a special case. Solution in terms of elliptic functions is possible for this, and also for the bead on a rotating hoop, the spherical pendulum, and the sphere rolling inside a sphere. There are also some special elementary function solutions, e.g. for the nonlinear pendulum,

$$\theta(t) = 2 \arctan(\sinh(t \sqrt{\frac{g}{l}})).$$
In Cartesian \((x, y, z)\)-coordinates, various motions look like ‘orbits’ precessing.

Poincare-Lindstedt, Fourier-series methods can be used to study some weakly nonlinear oscillations. We have done this for the bead on the rotating hoop, and also for the small oscillations about the stable equilibria associated with a disk rolling nearly flat to the table.

1.4 Lagrangians

Sadly, these are a bit intricate for non-holonomic systems. For the various pendula examples, they are easy to set up.

1.5 The rolling disk example

The disk rolling on a horizontal plane when the disk is nearly flat to the plane is studied because of its possible relevance to some aspects of ‘Euler’s disk’. For this problem we investigate, not only the linear stability of the equilibria, but the nonlinear orbits nearby.

The disk rolling inside a spherical shell seems to be a more advanced exercise. We have done no more than repeat stability calculations reported recently (and there are but minor differences in the results).

There have been Mathematica animations (and some theoretical study) of the straight-line motions of a rolling hoop with a point mass at its perimeter. (The Mathematica can be found, on the Web, amongst the examples in the ‘Interdisciplinary Lively Applications Project’, with title ‘The hopping hoop’. A few of the authors of papers are: Pritchett, Amer. Math Monthly 1999, Liu and Yun Acta Mechanica Sinica 2004.) The hoop can hop up from the table. The stability to small perturbations in 3 dimensions would involve Mathieu functions. I am not aware of where this has been treated, or, indeed, of where the relevant d.e.s are given.

2 The author’s motivations in these studies: the rolling disk example

In 2005, Prof Keith Moffatt passed through WA on his way to a meeting in Adelaide. During his visit to UWA he gave a colloquium on ‘The paradoxical behaviour of spinning bodies’. One of the mechanics demonstrations was ‘Euler’s disk’, another, the curious stability properties of the ‘tippe-top’. During a workshop series before his colloquium, the rolling disk d.e.s - a system of d.e.s which is 4th order - were studied and I observed that these were exactly integrable. Though this observation was new to myself, and to Keith Moffatt, I eventually discovered that it was reasonably well-known to some others, and had, in fact, been first noticed around 1900. I’m
not surprised that my observation proved to be a rediscovery, as this is an old, well-studied, problem. The sums, however, are pretty easy, as is the computer simulation work. Teaching is important. Good problems to use in it are worth exploring. So it seems reasonable to attempt to explain, in appropriate engineering or applied maths units, some of the work, e.g. where examples of stability of equilibria are required. Indeed, the stability of the equilibria for a disk rolling on a horizontal plane was treated in my 2nd year applied maths text, [SG], when I studied this, here at UWA, in 1965. Forty years later, it is easier, with Computer Algebra systems doing the routine calculation, and it is prettier, with computer graphics.

As stated earlier, numerical solution of o.d.e.s is easy, and the exact integrals - that the rolling disk system is integrable - can be used as a check on the numerics.

The commercial Euler disk is sold with a convex base. This motivates our study of the problem of a disk rolling inside a sphere. The equilibria, and their stability, can be studied in a similar way to rolling on a flat base. However, this system of d.e.s is 6th order, and is not integrable. The variety of behaviour it exhibits may well be richer than the case of rolling on the plane, but little seems to be known of this.

The studies associated with the disk rolling in a sphere are decidedly preliminary. The problems involving disks (or spheres) rolling are ‘non-holonomic’. Simpler, holonomic, problems that use similar mathematics, were listed earlier, e.g. the bead on a rotating hoop, the spherical pendulum, etc.. These simpler problems can be used to illustrate somewhat similar calculations on stability of equilibria. Also, the ‘precession of orbits’ in these simpler problems looks a bit like the trajectory of the centre of the rolling disk. However, this is best learnt by doing the sums or computer work, and that we treat in the later handouts in this series.

References