References

The following *Advanced Engineering Mathematics* books contain chapters on Laplace Transforms: HDR Chpt 9, ON Chpt 3. Matlab (in its Symbolic Toolbox) contains the functions `laplace` and `ilaplace` for the calculation, respectively, of Laplace transforms and their inverses. Tables of transforms (and there is one on Wikipedia) were used in earlier pre-Matlab times (and remain in use in hand calculation today – such as exam rooms).

1 Theory

1.1 Introduction

There is an account at

http://en.wikipedia.org/wiki/Laplace_transform

The wikipedia account has much more than in these notes.

Laplace transforms can be used to solve (some) linear DEs by:

- transforming the DE into a simpler problem,
- solving the simpler problem,
- transforming that solution back to obtain the solution of the original problem.

Laplace transforms are also used to solve (some) linear partial DEs but we only apply them to ordinary DEs now.
original (hard) problem \hspace{2cm} \text{soln to original problem}

\[
\mathcal{L} \quad \downarrow \quad \mathcal{L}^{-1}
\]

transformed (easy) problem \hspace{2cm} \text{soln to transformed problem}

(easy) solve

Figure 1: The main idea of transform methods.

1.2 Definition

Definition. (HDR §9.1; ON §3.1 Defn 3.1.) If \( f(t) \) is defined for all \( t \geq 0 \) then, if the integral
\[
F(s) = \int_0^\infty e^{-st}f(t)dt
\]
exists, the function \( F(s) \) is known as the Laplace transform of \( f(t) \). We often write \( F(s) \) as \( \mathcal{L}(f) \), or slightly more precisely as \( \mathcal{L}(f)(s) \).

Notice that \( F \) is a function of a new variable \( s \).

We also say \( f(t) \) is the inverse Laplace transform of \( F(s) \), written as \( f(t) = \mathcal{L}^{-1}(F) \).

The main idea and use of transforms

laplace (and fourier) transforms are used to transform hard, but linear, problems to easier problems.

<table>
<thead>
<tr>
<th>hard problem</th>
<th>linear o.d.e</th>
<th>p.d.e</th>
<th>linear o.d.e</th>
<th>some integral equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>easier problem</td>
<td>linear algebraic problem</td>
<td>algebraic problem</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In practice Matlab’s Symbolic Toolbox (and, before them, tables) are (were) used. In Student Matlab, the relevant commands are \texttt{laplace} and \texttt{ilaplace}.

Examples

1. If \( f(t) = 1 \) for \( t \geq 0 \) we obtain
\[
F(s) = \int_0^\infty e^{-st}dt = \left[ -\frac{e^{-st}}{s} \right]_0^\infty
\]
and this integral exists for \( s > 0 \), giving that
\[
\mathcal{L}(1) = \frac{1}{s}.
\]

Notice that here \( F \) is not defined for all real values of \( s \), just for \( s > 0 \).

2. Consider \( f(t) = e^{at} \) for \( t \geq 0 \), where \( a \) is a constant, then
\[
F(s) = \int_0^\infty e^{-st} e^{at} \, dt = e^{(a-s)t} \bigg|_0^\infty = \left[ \frac{e^{(a-s)t}}{a-s} \right]_0^\infty
\]
and this integral exists for \( s > a \), so that
\[
\mathcal{L}(e^{at}) = \frac{1}{s-a} \text{ for } s > a.
\]

These are just two examples of Laplace transforms, but to obtain others (besides using Matlab’s Symbolic Toolbox) we can use some of the simple properties of the Laplace transform operation.

### 1.3 Linearity of Laplace transforms
The following property is an immediate consequence of the definition of Laplace transform.

**Theorem 1** If the Laplace transforms of two functions \( f(t) \) and \( g(t) \) exist then for any constants \( a \) and \( b \),
\[
\mathcal{L} (af(t) + bg(t)) = a\mathcal{L} (f(t)) + b\mathcal{L} (g(t)).
\]
See HDR Thm 9.1; ON Thm 3.1.
Examples

1. For \( f(t) = \cosh(at) = \frac{1}{2}(e^{at} + e^{-at}) \) we can use

\[
\mathcal{L}(\cosh(at)) = \frac{1}{2} \left( \mathcal{L}(e^{at}) + \mathcal{L}(e^{-at}) \right) \\
= \frac{1}{2} \left( \frac{1}{s-a} + \frac{1}{s+a} \right) \\
= \frac{s}{s^2 - a^2}
\]

provided both \( s > a \) and \( s > -a \), i.e., \( s > |a| \).

2. The same linearity property can be used for the inverse transform. For

\[
F(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s},
\]

we get

\[
\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left( \frac{1}{s(s-1)} \right) = \mathcal{L}^{-1}\left( \frac{1}{s-1} - \frac{1}{s} \right)
\]

so that

\[
\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left( \frac{1}{s-1} \right) - \mathcal{L}^{-1}\left( \frac{1}{s} \right) = e^t - 1.
\]

There will be more in the subsection 1.5, where you will learn a systematic technique using partial fractions.

1.4 Existence of Laplace transforms

A function is said to be **piecewise continuous** on an interval \([a, b]\) if it is continuous except at a finite number of finite jumps. See Figure 2. [http://en.wikipedia.org/wiki/Piecewise](http://en.wikipedia.org/wiki/Piecewise)

If \( f \) is piecewise continuous on every interval \([0, T]\) with \( T > 0 \) we say it is of **exponential order** if it satisfies, for some constants \( \gamma \) and \( M \),

\[
|f(t)| \leq Me^{\gamma t} \text{ for all } t \geq 0.
\]

When this holds, we say that the **exponential order of** \( f \) is \( \leq \gamma \).
Theorem 2  If $f(t)$ is piecewise continuous on every interval $[0, T]$ with $T > 0$ and $f$ is of exponential order $\leq \gamma$, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > \gamma$.

This is a sufficient condition for $\mathcal{L}(f)$ to exist, but not a necessary condition.

If the Laplace transform of a given function exists then it is unique. Conversely, if two functions have the same Laplace transform then they can differ at only isolated points.

Examples

1. For $f(t) = 1$ then

   $$|f(t)| \leq 1 \ (= 1e^{0t}) \text{ for } t \geq 0$$

   so that $M = 1$ and $\gamma = 0$ are sufficient. So the transform exists for $s > 0$, as we noted earlier.

2. For $f(t) = e^{at}$ then $M = 1$ and $\gamma = a$ can be used to show $|f(t)| \leq Me^{\gamma t}$ for all $t \geq 0$ and so, as noted earlier, the transform exists for $s > a$.

3. For $f(t) = \cosh(at)$ then

   $$|f(t)| \leq \frac{1}{2}(e^{|a|t} + 1) \leq e^{|a|t} \text{ for } t \geq 0$$
so that $M = 1$ and $\gamma = |a|$, and the transform exists for $s > |a|$ as before.

4. For $f(t) = e^{t^2}$ there are no $M$ and $\gamma$ for which

$$e^{t^2} \leq Me^{\gamma t} \text{ for all } t \geq 0$$

since for any $M$ and $\gamma$ we can find a $t_0$ for which it is false for all $t > t_0$. The integral for $L(e^{t^2})$ does not exist in this case.

5. For $f(t) = t^n$ for some integer $n \geq 0$ then, from the Taylor series for $e^t$,

$$\frac{t^n}{n!} \leq 1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} \leq e^t \text{ for } t \geq 0$$

and hence $M = n!$ and $\gamma = 1$ are sufficient to verify that $L(t^n)$ exists for $s > 1$.

1.4.1 The Gamma function $\Gamma(x)$

$\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.$$ 

In Matlab this is written $\text{gamma}(x)$.

(a) For any fixed $x$, we have $t^{x-1}e^{-t/2} \leq 1$ for all sufficiently large $t > 0$. Thus, for such $t$ we have $t^{x-1}e^{-t} \leq e^{-t/2}$. Since $\int_0^\infty e^{-t/2} \, dt$ is convergent (exercise), it follows that the integral defining $\Gamma(x)$ is convergent, too. That is, $\Gamma(x)$ is well-defined for $x > 0$.

(b) We have

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = \lim_{t \to \infty} e^{-t} = 1.$$
(c) For any $x > 0$ we have

$$\Gamma(x + 1) = x\Gamma(x).$$

Indeed, using integration by parts, we have

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} \, dt = -\int_0^\infty t^x (e^{-t})' \, dt$$

$$= -[t^x e^{-t}]_0^\infty + x \int_0^\infty t^{x-1} e^{-t} \, dt = -\lim_{t \to \infty} t^x e^{-t} + x\Gamma(x) = x\Gamma(x).$$

(d) From (b) and (c) one derives that $\Gamma(n + 1) = n!$ for every integer $n \geq 0$.

It can be shown that $\Gamma(x)$ has derivatives of all orders (and a variety of other remarkable properties). A plot of its graph is given in figure 3.

![Figure 3: Plot of $\Gamma(x)$](image)
1.4.2 The Laplace transform of $t^\alpha$

In connection with this last example, where we showed that the Laplace transform of $t^n$ would exist, we can proceed to actually calculate its Laplace transform:

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n dt$$

Substituting $x = ts$ gives that

$$\mathcal{L}(t^n) = \int_0^\infty e^{-x} \left( \frac{x}{s} \right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

from the definition of the Gamma function. This exists for all $s > 0$, not just $s > 1$.

More generally, for any real number $\alpha > -1$ we can show similarly that

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}$$

for $s > 0$.

1.5 Inverse Laplace transforms of rational functions

The inverse Laplace transform is also a linear operator.

$$\mathcal{L}^{-1}(aF + bG) = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G).$$

We can use this, and partial fraction decompositions, to find the inverse Laplace transforms of rational functions.

If the Laplace transform $F(s)$ of some function $f(t)$ has the form

$$F(s) = \frac{P(s)}{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials, then we can find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}(F)$ using partial fractions. Partial fractions are also used as part of the algorithm for integration of rational functions.

Here are the inverse Laplace transforms of some basic rational functions. From before we have:

$$\mathcal{L}^{-1} \left( \frac{1}{s + a} \right) = e^{-at}.$$
We also have
\[ \mathcal{L}^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin(at), \quad \mathcal{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos(at). \]

Apart from this, we also have the following two formulae (cf. Example in subsection 1.3) which are sometimes useful
\[ \mathcal{L}^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{1}{a} \sinh(at), \quad \mathcal{L}^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh(at). \]

**Remark.** Later, we will see a formula for \( \mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right) \).

To recall the method of partial fractions and demonstrate how it works with the inverse Laplace transform, we will consider two examples.

**Examples:**

1. Suppose \( F(s) = \frac{2s - 1}{(s - 1)(s + 3)} \), \( s > 1 \), and we want to find \( f(t) = \mathcal{L}^{-1}(F) \). First, using partial fractions, we write
\[
F(s) = \frac{2s - 1}{(s - 1)(s + 3)} = \frac{A}{s - 1} + \frac{B}{s + 1} + \frac{C}{s + 3}.
\]

This is equivalent to
\[
2s - 1 = A(s + 1)(s + 3) + B(s - 1)(s + 3) + C(s - 1)(s + 1).
\]

From this with \( s = 1 \) we get \( A = 1/8 \). Similarly, \( s = -1 \) gives \( B = 3/4 \), while \( s = -3 \) implies \( C = -7/8 \). Thus,
\[
F(s) = \frac{1}{8(s - 1)} + \frac{3}{4(s + 1)} - \frac{7}{8(s + 3)},
\]

and therefore
\[
f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{8} \mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) + \frac{3}{4} \mathcal{L}^{-1}\left(\frac{1}{s + 1}\right) - \frac{7}{8} \mathcal{L}^{-1}\left(\frac{1}{s + 3}\right) = \frac{1}{8} e^t + \frac{3}{4} e^{-t} - \frac{7}{8} e^{-3t}.
\]
2. Suppose $F(s) = \frac{2s^2 - s + 4}{s^3 + 4s}$. To find $f(t) = \mathcal{L}^{-1}(F)$, we first use partial fractions:

$$F(s) = \frac{2s^2 - s + 4}{s^3 + 4s} = \frac{2s^2 - s + 4}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}.$$  

This is equivalent to

$$2s^2 - s + 4 = A(s^2 + 4) + (Bs + C)s = (A + B)s^2 + Cs + 4A,$$

so we must have $A + B = 2$, $C = -1$ and $4A = 4$. This gives $A = 1$, $B = 1$, $C = -1$. Thus,

$$F(s) = \frac{1}{s} + \frac{s - 1}{s^2 + 4} = \frac{1}{s} + \frac{s}{s^2 + 4} - \frac{1}{s^2 + 4},$$

and therefore

$$f(t) = \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2 + 4}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2 + 4}\right) = 1 + \cos(2t) - \frac{1}{2} \sin(2t).$$

### 1.6 Transforms of derivatives

This is a crucial property of Laplace transforms. Calculus operations on $f$ are replaced by algebraic operations on $\mathcal{L}(f)$.

**Theorem 3** If $f(t)$ is continuous and of exponential order $\leq \gamma$, and if $f'(t)$ exists and is piecewise continuous over $[0, T]$ for all $T \geq 0$ then the Laplace transform of $f'(t)$ exists for $s > \gamma$ and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

**Proof.** This is easy to verify using integration by parts

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t) dt = \left[ e^{-st} f(t) \right]_0^\infty + s \int_0^\infty e^{-st} f(t) dt = -f(0) + s\mathcal{L}(f)$$

since

$$|e^{-st} f(t)| \leq Me^{(\gamma-s)t} \to 0 \text{ as } t \to \infty.$$
when $s > \gamma$.

The same process can be used to obtain Laplace transforms of higher derivatives of $f$, for example

$$L(f'') = sL(f') - f'(0) = s[sL(f) - f(0)] - f'(0)$$

so that

$$L(f') = s^2L(f) - sf(0) - f'(0).$$

Similarly,

$$L(f'''') = s^3L(f) - s^2f(0) - sf'(0) - f''(0)$$

and, most generally (HDR equation (9.26); ON Thm 3.6),

$$L(f^{(n)}) = s^nL(f) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0).$$

Examples

1. For $f(t) = \sinh(at)$ then $f'(t) = a\cosh(at)$ and using the transform of $\cosh(at)$ from earlier in

$$aL(\cosh(at)) = sL(\sinh(at)) - \sinh(0)$$

gives that

$$L(\sinh(at)) = \frac{a}{s}L(\cosh(at)) = \frac{a}{s} \left( \frac{s}{s^2 - a^2} \right) = \frac{a}{s^2 - a^2}.$$  

2. A similar method can be used to find $L(\sin(\omega t))$. If $f(t) = \sin(\omega t)$ then $f''(t) = -\omega^2 f(t)$ and so

$$L(f'') = -\omega^2L(f) = s^2L(f) - sf(0) - f'(0).$$

Collecting the two terms involving $L(f)$,

$$(\omega^2 + s^2)L(f) = sf(0) + f'(0)$$

and, using that $f(0) = 0$ and $f'(0) = \omega$ we obtain

$$L(f) = \frac{\omega}{s^2 + \omega^2}.$$
This is valid for \( s > 0 \), since \(|f(t)| \leq 1\) for all \( t \).

Similarly, for \( f(t) = \cos(\omega t) \), where \( f(0) = 1 \) and \( f'(0) = 0 \), we obtain that
\[
\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2} \quad \text{for} \quad s > 0.
\]
The same technique can also be used to show that
\[
\mathcal{L}(t \sin(\omega t)) = \frac{2\omega s}{(s^2 + \omega^2)^2}
\]
and so on . . .

Using all of the techniques above (and some more to come later) we can build up a table of Laplace transforms of frequently-encountered functions.

### 1.7 Solving differential equations

Laplace transforms can be applied to initial-value problems for constant coefficient ordinary DEs by reducing them to solving an algebraic equation.

For example, we can seek a solution \( y(t) \) of
\[
y'' + ay' + by = r(t) \quad \text{for} \quad t \geq 0,
\]
where \( a, b \) are constant and \( r(t) \) is a given function, such that \( y \) satisfies the initial conditions
\[
y(0) = K_0 \quad \text{and} \quad y'(0) = K_1.
\]
To solve this, we first transform the DE, writing
\[
\mathcal{L}(y'') + a\mathcal{L}(y') + b\mathcal{L}(y) = R(s),
\]
where \( R(s) = \mathcal{L}(r) \). In terms of \( Y(s) = \mathcal{L}(y) \), this gives the subsidiary equations
\[
(s^2 Y - sy(0) - y'(0)) + a(sY - y(0)) + bY = R(s)
\]
and hence \( Y \) can be found from
\[
(s^2 + as + b)Y(s) = R(s) + (s + a)y(0) + y'(0).
\]
Once $Y(s)$ has been determined we can invert the transform to yield the solution $y(t) = \mathcal{L}^{-1}(Y)$.

**Example**

For the problem

$$y'' - y = t \text{ with } y(0) = 1 \text{ and } y'(0) = 1$$

the subsidiary equation is

$$(s^2 Y(s) - s - 1) - Y = \frac{1}{s^2}$$

and hence

$$(s^2 - 1)Y = s + 1 + \frac{1}{s^2}.$$  

It follows that

$$Y(s) = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)} = \frac{1}{s - 1} + \frac{1}{s^2 - 1} - \frac{1}{s^2},$$

using partial fractions and so, from the table of Laplace transforms earlier,

$$y(t) = e^t + \sinh(t) - t.$$  

An advantage of this technique is that it is not necessary to solve for the general solution of the homogeneous DE and then determine the arbitrary constants in that solution.

### 1.8 Transforms of integrals

**Theorem 4** If $f(t)$ is piecewise continuous and of exponential order $\leq \gamma$, then for $s > \gamma$ and $s \neq 0$ we have

$$\mathcal{L}\left(\int_0^t f(u) \, du\right) = \frac{1}{s} \mathcal{L}(f)(s).$$

That is, if the Laplace transform of $f(t)$ is $F(s)$ and $g(t) = \int_0^t f(u) \, du$, then

$$\mathcal{L}(g)(s) = \frac{F(s)}{s} \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t \mathcal{L}^{-1}(F)(u) \, du.$$
Proof. Denote the Laplace transform of $g(t)$ by $G(s)$. Since $g'(t) = f(t)$ and $g(0) = 0$, Theorem 3 implies (for $s > \gamma$)

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}(g)(s) = sL(g)(s) - g(0) = sG(s).$$

Thus, for $s > \gamma$, $s \neq 0$, we have $G(s) = F(s)/s$. This can be useful to help determine the inverse transform of functions which have $s$ appearing in the denominator.

**Example**

If $\mathcal{L}(f) = \frac{1}{s(s^2+\omega^2)}$ and we have that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 + \omega^2}\right) = \frac{1}{\omega}\sin(\omega t)$$

then the formula above tells us that

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2 + \omega^2)}\right) = \frac{1}{\omega} \int_0^t \sin(\omega t) dt = \frac{1}{\omega^2}(1 - \cos(\omega t)).$$

1.9 ‘s-shifting’

(See HDR Thm 9.2; ON §3.3)

**Theorem 5** If $F(s)$ is the Laplace transform of $f(t)$ for $s > b \geq 0$, then the Laplace transform of $e^{at}f(t)$ is

$$\mathcal{L}\left(e^{at}f(t)\right) = F(s - a).$$

for $s - a > b$. In other words,

$$\mathcal{L}^{-1}(F(s - a)) = e^{at}f(t).$$

(See HDR Thm 9.2; ON Thm 3.8.)

**Proof.** $F(s) = \mathcal{L}(f)$ and

$$\mathcal{L}\left(e^{at}f(t)\right) = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = F(s - a),$$

which proves the statement.
Examples

1. The transform of \( f(t) = e^{at}t^n \) is

\[
\mathcal{L}(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a.
\]

2. The transform of \( f(t) = e^{at}\cos(\omega t) \) is

\[
\mathcal{L} \left( e^{at}\cos(\omega t) \right) = \frac{s-a}{(s-a)^2 + \omega^2} \quad \text{for } s > a.
\]

1.10 ‘t-shifting’

Performing a ‘shift on the \( t \) axis’ for \( f(t) \) can have a similar type of effect on its Laplace transform.

If \( f(t) \) has transform \( F(s) \) then the function

\[
g(t) = \begin{cases} 
0 & \text{if } t < a \\
f(t-a) & \text{if } t > a
\end{cases}
\]

has transform \( \mathcal{L}(g) = e^{-as}F(s) \) for any \( a \geq 0 \).

Another way to write this is to use the Unit Step Function \( u(t-a) \) defined by

\[
u(t-a) = \begin{cases} 
0 & \text{if } t < a \\
1 & \text{if } t > a
\end{cases}
\]

which is shown in Figure 4.

(Taken from http://en.wikipedia.org/wiki/Heaviside_step_function) \( u(t-a) \) is also written \( H(t-a) \) or heaviside\((t-a)\) or UnitStep\((t-a)\). See HDR p420; ON §3.3, Definition 3.2.

Theorem 6 If the Laplace transform of \( f(t) \) is \( F(s) \) for \( s > b \), then for any \( a \geq 0 \) we have

\[
\mathcal{L} \left( f(t-a)H(t-a) \right) = e^{-as}F(s)
\]

for \( s > b \). Consequently,

\[
\mathcal{L}^{-1} \left( e^{-as}F(s) \right) = f(t-a)H(t-a).
\]

(See HDR Thm 9.3; ON Thm 3.10.)

Note that these results are not valid if \( a \) is negative.
Examples

1. If \( f(t) = 1 \) for \( t > 0 \) then we can determine the Laplace transform of \( u(t - a) \) for any \( a \geq 0 \), using that

\[
\mathcal{L}(u(t - a)) = \mathcal{L}(f(t - a)u(t - a)) = e^{-as}F(s).
\]

Here \( F(s) = \mathcal{L}(f) = \frac{1}{s} \) from earlier, so that

\[
\mathcal{L}(u(t - a)) = \frac{e^{-as}}{s}.
\]

This result can also be proven directly from the definition.

2. To determine the inverse transform of \( e^{-4s}/s^3 \) we first recognise that

\[
\mathcal{L}^{-1}\left(1/s^3\right) = t^2/2
\]

so that, with \( a = 4 \),

\[
\mathcal{L}^{-1}\left(e^{-4s}/s^3\right) = \begin{cases} 0 & \text{if } t < 4 \\ (t - 4)^2/2 & \text{if } t > 4. \end{cases}
\]

3. Consider the differential equation

\[
y'' + y = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases} = u(t - 1) - u(t - 2),
\]

Figure 4: heaviside function.
with initial conditions $y(0) = 0$ and $y'(0) = 1$. Taking transforms, $Y(s) = \mathcal{L}(y)$ satisfies

$$s^2 Y - sy(0) - y'(0) + Y = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$  

and solving for $Y$ yields that

$$Y(s) = \frac{1}{s^2 + 1} + \frac{(e^{-s} - e^{-2s})}{s(s^2 + 1)} \frac{1}{s^2 + 1}$$

$$= \frac{1}{s^2 + 1} + \frac{(e^{-s} - e^{-2s})}{s} \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right).$$

Inverting the transforms we get

$$y(t) = \sin(t) + u(t - 1) (1 - \cos(t - 1)) - u(t - 2) (1 - \cos(t - 2)),$$

so that $y$ is equal to

- $\sin(t)$ for $0 < t < 1$
- $\sin(t) + 1 - \cos(t - 1)$ for $1 < t < 2$
- $\sin(t) - \cos(t - 1) + \cos(t - 2)$ for $t > 2.$

Note that in this solution both $y$ and $y'$ are continuous at $t = 1$ and $t = 2$.

### 1.11 Some more d.e. applications

![RC-circuit diagram](image)

Figure 5: An RC-circuit with input $v(t)$. 

17
1.11.1 RC-circuit

The current in the RC circuit depicted in Fig. 5 is governed by the equation

\[ Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = v(t). \]

Let us consider a single square wave input for \( v(t) \),

\[ v(t) = \begin{cases} 
0 & \text{if } t < a, \\
V_0 & \text{if } a \leq t \leq b, \\
0 & \text{if } b < t.
\end{cases} \]

which can also be written as

\[ v(t) = V_0 \left( u(t - a) - u(t - b) \right). \]

So that the Laplace transform of the input signal is given by

\[ V(s) = \frac{V_0}{s} \left( e^{-as} - e^{-bs} \right). \]

The subsidiary equation for the system can then be written as

\[ RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s} \left( e^{-as} - e^{-bs} \right), \]

or,

\[ \frac{(RCs + 1)}{sC} I(s) = \frac{V_0}{s} \left( e^{-as} - e^{-bs} \right). \]

So the Laplace transform of the current can be written as

\[ I(s) = F(s) \left( e^{-as} - e^{-bs} \right), \]

with

\[ F(s) = \frac{V_0/R}{s + 1/RC} \]

which has the inverse Laplace transform

\[ \mathcal{L}^{-1}(F(s)) = \frac{V_0}{R} e^{-t/(RC)}. \]

Shifting in \( t \) then yields the solution for the current:

\[ i(t) = \mathcal{L}^{-1} \left( e^{-as} F(s) \right) - \mathcal{L}^{-1} \left( e^{-bs} F(s) \right), \]

\[ = \frac{V_0}{R} \left( u(t - a)e^{-a(t-a)/(RC)} - u(t - b)e^{-b(t-b)/(RC)} \right). \]

This resulting current is depicted in Fig. 6.
Figure 6: The current in an RC circuit with a single square wave voltage input. $a = 1$, $b = 2$, $R = 10\Omega$, $C = 0.1$ Farad and $V_01$ Volt.
1.11.2 Response of an undamped system

Let us consider a system described by the following differential equation:

\[ y'' + 2y = r(t), \]

with initial values \( y(0) = 0 \) and \( y'(0) = 0 \). The Laplace transform of the single square wave \( r(t) = u(t) - u(t - 1) \) is given by

\[ R(s) = \frac{1}{s} - \frac{e^{-s}}{s}. \]

The subsidiary equation then reads

\[ (s^2 + 2)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}, \]

so that

\[ Y(s) = \frac{1}{s(s^2 + 2)} - \frac{e^{-s}}{s(s^2 + 2)}. \]

Using partial fractions, we can write

\[ \frac{1}{s(s^2 + 2)} = \frac{1}{2} \left( \frac{1}{s} - \frac{s}{s^2 + 2} \right), \]

so that the inverse Laplace transform can be found as

\[ y(t) = \frac{1}{2} \left( 1 - \cos(\sqrt{2} t) \right) - \frac{1}{2} u(t - 1) \left( 1 - \cos(\sqrt{2} (t - 1)) \right) \]

which can also be written as

\[ y(t) = \begin{cases} 
\frac{1}{2} - \frac{1}{2} \cos(\sqrt{2} t), & \text{if } 0 \leq t < 1, \\
\frac{1}{2} \cos(\sqrt{2} (t - 1)) - \frac{1}{2} \cos(\sqrt{2} t), & \text{if } t \geq 1.
\end{cases} \]

The response \( y(t) \) is pictured in Fig. 7.
Figure 7: The response of an undamped system to a single square wave.
1.11.3 A damped vibrating system

Let us calculate the response of the undamped vibrating system governed by the equation

$$y'' + 3y' + 2y = r(t),$$

and the initial values $y(0) = 0$ and $y'(0) = 0$ to a single square wave input signal, $r(t) = u(t) - u(t - 1)$.

The subsidiary equation is

$$(s^2 + 3s + 2)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s},$$

so that the Laplace transform of $y(t)$ can be written as

$$Y(s) = F(s)(1 - e^{-s}),$$

with

$$F(s) = \frac{1}{s(s^2 + 3s + 2)} = \frac{1}{s(s + 2)(s + 1)} = \frac{1}{2s} - \frac{1}{s + 1} + \frac{1}{2(s + 2)}.$$  

The inverse Laplace transform of $F(s)$ is then

$$f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$  

The response $y(t)$ can then be written as

$$y(t) = f(t) - u(t - 1)f(t - 1)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u(t - 1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right).$$

We can write $y(t)$ also as

$$y(t) = \begin{cases} 
\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} & \text{if } 0 \leq t < 1, \\
(e - 1)e^{-t} + \frac{(1-e^2)e^{-2t}}{2} & \text{if } 1 \leq t.
\end{cases}$$

The response $y(t)$ is depicted in Fig. 8.
Figure 8: The response of a damped vibrating system to a single square wave.

1.12 Derivatives of transforms

**Theorem 7** Let \( f(t) \) be of exponential order \( \leq \gamma \) and piecewise continuous on \([0, T]\) for any \( T > 0 \): then, as in Theorem 2,

\[
F(s) = \mathcal{L}(f)(s) = \int_{0}^{\infty} e^{-st} f(t) dt
\]

exists for \( s > \gamma \). Moreover, we have, for \( s > \gamma \),

\[
F'(s) = -\int_{0}^{\infty} e^{-st} tf(t) dt.
\]

Consequently,

\[
\mathcal{L}^{-1} (F'(s)) = -tf(t)
\]

and that

\[
\mathcal{L} (tf(t)) = -F'(s).
\]

One can use Theorem 7 to find more function-transform pairs.

**Example**

Earlier, the transform of \( f(t) = t \sin(\omega t) \) was found by differentiating \( f \) twice.
We now have an easier way since
\[
\mathcal{L}(t \sin(\omega t)) = -\frac{d}{ds} \left( \frac{\omega}{s^2 + \omega^2} \right)
= \frac{2\omega}{(s^2 + \omega^2)^2}.
\]

### 1.13 Integrals of transforms

**Theorem 8** Let \( f(t) \) satisfy the existence conditions of the first sentence of Theorem 7. If the limit of \( f(t)/t \) exists as \( t \to 0 \) from above then

\[
\mathcal{L} \left( \frac{f(t)}{t} \right) = \int_{s}^{\infty} s F(\hat{\gamma}) d\hat{\gamma},
\]

for \( s > \gamma \), so that

\[
\mathcal{L}^{-1} \left( \int_{s}^{\infty} s F(\hat{\gamma}) d\hat{\gamma} \right) = \frac{f(t)}{t}.
\]

This is quite easy to show, provided we can swap the order of integration in the expression

\[
\int_{s}^{\infty} F(\hat{\gamma}) d\hat{\gamma} = \int_{s}^{\infty} \left( \int_{0}^{\infty} e^{-\hat{\gamma}t} f(t) dt \right) d\hat{\gamma},
\]

which is allowable under the conditions above, so

\[
\int_{s}^{\infty} F(\hat{\gamma}) d\hat{\gamma} = \int_{0}^{\infty} \left( \int_{s}^{\infty} e^{-\hat{\gamma}t} d\hat{\gamma} \right) f(t) dt
= \int_{0}^{\infty} \left[ \frac{e^{-\hat{\gamma}t}}{t} \right]_{s}^{\infty} f(t) dt
= \int_{0}^{\infty} \frac{e^{-\hat{\gamma}t}}{t} f(t) dt = \mathcal{L} \left( \frac{f(t)}{t} \right).
\]

**Examples**

1. For \( f(t) = \frac{\sin(\omega t)}{t} \) we have \( \lim_{t \to 0^+} f(t) = \omega \) and hence we can evaluate the transform of \( f \) using

\[
\mathcal{L} \left( \frac{\sin(\omega t)}{t} \right) = \int_{s}^{\infty} \frac{\omega}{s^2 + \omega^2} d\hat{\gamma}
\]

24
\[ = \left[ \arctan \left( \frac{s}{\omega} \right) \right]_{s}^{\infty} = \frac{\pi}{2} - \arctan \left( \frac{s}{\omega} \right) = \arctan \left( \frac{s}{\omega} \right). \]

2. To evaluate the inverse transform of

\[ F(s) = \ln \left( 1 + \frac{1}{s^2} \right) \]

we first note that \( F(s) \to 0 \) as \( s \to \infty \). Then, using that

\[ F(s) = \ln(s^2 + 1) - \ln(s^2), \]

we obtain

\[ F(s) = -\int_{s}^{\infty} \left( \frac{2s}{s^2 + 1} - \frac{2}{s} \right) d\hat{s} \]

and, from the table earlier, the integrand is the transform of \( f(t) = -(2 \cos(t) - 2) \).

This satisfies the required conditions on \( f \), so we can conclude that

\[ \mathcal{L}^{-1}(F) = \frac{2}{t}(1 - \cos(t)). \]

1.14 Convolution

(HDR §9.5; ON §3.4)

Sometimes we want to invert a transform which is a product of two known transforms.

Given two functions \( f \) and \( g \), define the convolution \((f \otimes g)\) of \( f \) and \( g \) by

\[ (f \otimes g)(t) = \int_{0}^{t} f(\tau)g(t - \tau)d\tau. \]

(HDR equation (9.51); ON §3.4 Definition 3.3.)

Theorem 9 (The Convolution Theorem) Let \( f \) and \( g \) be functions having Laplace Transforms, and denote these by \( F \) and \( G \), both defined for \( s > \gamma \). Then

\[ \mathcal{L}((f \otimes g)) = F(s)G(s) \quad (s > \gamma). \]
(See HDR Thm 9.8; ON Thm 3.11.)

We defer the proof until after some examples.

**Examples**

1. Consider the two functions \( f(t) = t \) and \( g(t) = t \). Their convolution is

\[
t \otimes t = \int_0^t \tau(t - \tau)d\tau = \left[ \frac{\tau^2}{2} t - \frac{\tau^3}{3} \right]_0^t = \frac{t^3}{6}.
\]

As shown earlier, the transform of \( t^n \) is \( n!/s^{n+1} \) so the transform of \( t \otimes t \) is \( 1/s^4 \), which is the product of the transforms \( 1/s^2 \) of \( f \) and \( g \).

2. To find the inverse transform of \( H(s) = \frac{1}{(s^2 + 1)^2} \) we use that

\[
\mathcal{L}(\sin(t)) = \frac{1}{s^2 + 1}
\]

and then

\[
h(t) = \mathcal{L}^{-1}(H) = \sin(t) \otimes \sin(t) = \int_0^t \sin(\tau) \sin(t - \tau)d\tau.
\]

Evaluating this by Matlab’s Symbolic Toolbox or by using cosine sum formulae gives that the right-hand side above is equal to

\[
\frac{1}{2} \int_0^t \left( \cos(\tau - (t - \tau)) - \cos(\tau + (t - \tau)) \right) d\tau
\]

and hence

\[
h(t) = \frac{1}{2} \int_0^t \left( \cos(2\tau - t) - \cos(t) \right) d\tau
\]

\[
= \frac{1}{2} \left[ \frac{1}{2} \sin(2\tau - t) - \tau \cos(t) \right]_0^t
\]

\[
= \frac{1}{2} \sin(t) - \frac{1}{2} t \cos(t).
\]

From above, we then have that

\[
\mathcal{L}^{-1}\left( \frac{1}{(s^2 + 1)^2} \right) = \frac{1}{2} \sin(t) - \frac{1}{2} t \cos(t),
\]
and in the same manner it can be shown that

\[ \mathcal{L}^{-1}\left( \frac{s^2}{(s^2 + 1)^2} \right) = \frac{1}{2} \sin(t) + \frac{1}{2} t \cos(t) . \]

**Proof of The Convolution Theorem.** Taking the transform of \( h(t) = (f \otimes g)(t) \) gives

\[
H(s) = \int_0^\infty e^{-st} \left( \int_0^t f(\tau)g(t-\tau) \, d\tau \right) \, dt \\
= \int_0^\infty \int_0^t e^{-s\tau} f(\tau)e^{-s(t-\tau)}g(t-\tau) \, d\tau \, dt .
\]

The integration boundaries are determined by the region of integration in the \( t\tau \)-plane which is plotted in Fig. 9

![Figure 9: The region of integration in the convolution integral.](image)

Changing the order of integration of \( t \) and \( \tau \) over the region of integration from

\[
0 \leq \tau \leq t, \quad 0 \leq t < \infty
\]

to

\[
\tau \leq t < \infty, \quad 0 \leq \tau < \infty
\]

gives the integral

\[
H(s) = \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)e^{-s(t-\tau)}g(t-\tau) \, dt \, d\tau .
\]
Writing \( t' = t - \tau \) this is equivalent to

\[
H(s) = \int_0^\infty \int_0^\infty e^{-st} f(\tau)e^{-st'} g(t')dt'd\tau
= \left( \int_0^\infty e^{-st} f(\tau)d\tau \right) \left( \int_0^\infty e^{-st'} g(t')dt' \right)
= F(s)G(s).
\]

Properties of the convolution process include (see HDR §9.5; ON Thm 3.13):

\[
\begin{align*}
    f \otimes g &= g \otimes f & \text{commutative} \\
    f \otimes (g_1 + g_2) &= f \otimes g_1 + f \otimes g_2 & \text{distributive} \\
    (f \otimes g) \otimes h &= f \otimes (g \otimes h) & \text{associative} \\
    f \otimes 0 &= 0 \otimes f = 0
\end{align*}
\]

However, note that \( f \otimes 1 \) is not equal to \( f \) in general and that \( f \otimes f \) can be negative.

### 1.15 Application of the convolution to DEs

We saw earlier that the DE

\[
y'' + ay' + by = r(t) \quad \text{for} \quad t \geq 0,
\]

can be solved using Laplace transforms. If we write

\[
Q(s) = \frac{1}{s^2 + as + b}
\]

then the solution \( y(t) \) satisfying \( y(0) = y'(0) = 0 \) has transform \( Y(s) = R(s)Q(s) \), where \( R(s) \) is the transform of \( r(t) \).

Using the convolution property, it follows that

\[
y(t) = \int_0^t r(\tau)q(t - \tau)d\tau
\]

where \( q(t) \) is the transform of \( Q(s) \) above.
To solve this type of equation efficiently, it is also useful to know a few tricks about *partial fraction* expansions. If a transform $Y(s)$ has the form

$$Y(s) = \frac{F(s)}{G(s)},$$

where $F(s)$ and $G(s)$ are polynomials, then we can invert this easily.

If the degree of $F$ is less than the degree of $G$ and $G(s) = 0$ has simple (distinct) real roots then

$$Y(s) = \frac{F(s_1)}{G'(s_1)(s - s_1)} + \cdots + \frac{F(s_n)}{G'(s_n)(s - s_n)},$$

where $s_1, \ldots, s_n$ are all of the roots of $G(s) = 0$.

For example, when $F(s) = 1$ and $G(s) = s^2 + as + b$ then $G'(s) = 2s + a$ and both roots of $G(s) = 0$ are real and distinct if $\Delta = a^2 - 4b > 0$, so that

$$Q(s) = \frac{1}{(2s_1 + a)(s - s_1)} + \frac{1}{(2s_2 + a)(s - s_2)},$$

where $s_1 = \frac{1}{2}(-a + \sqrt{\Delta})$ and $s_2 = \frac{1}{2}(-a - \sqrt{\Delta})$. It follows that $q(t)$ used in the convolution earlier is

$$q(t) = \frac{1}{\sqrt{\Delta}} (e^{s_1 t} - e^{s_2 t}).$$

### 1.16 Application to integral equations

The convolution property can also be used to solve some linear *integral equations*, an integral equation being an equation where the function for which we must solve also appears under an integral sign.

For example, for the integral equation

$$y(t) = \sin(t) + \int_0^t y(\tau) \sin(t - \tau) d\tau$$

the transform $Y(s)$ of $y(t)$ satisfies

$$Y(s) = \frac{1}{s^2 + 1} + \frac{Y(s)}{s^2 + 1}.$$
It follows that $Y(s) = 1/s^2$ and hence $y(t) = t$.

Of course, this only works for integral equations in the form of a convolution.

We have now used Laplace transforms to solve both ordinary differential and integral equations. In practice, Laplace transforms are frequently used to solve partial differential equations (which often cannot be solved explicitly in other ways).

### 1.17 Transforms of periodic functions

HDR p441; ON §3.2 Problem 24.

**Theorem 10 (Transform of a periodic function.)** Let $f$ be piecewise continuous on $[0, \infty)$ and of exponential order. If $f$ is periodic with period $T$, then

$$
\mathcal{L}(f(t))(s) = \frac{1}{1 - \exp(-sT)} \int_0^T \exp(-st)f(t) \, dt.
$$

**Proof.**

$$
\mathcal{L}(f(t))(s) = \int_0^T \exp(-st)f(t) \, dt + \int_T^\infty \exp(-st)f(t) \, dt.
$$

Set $t = u + T$ in the second integral. Then (using periodicity of $f$)

$$
\begin{align*}
\mathcal{L}(f(t))(s) &= \int_0^T \exp(-st)f(t) \, dt + \exp(-sT) \int_0^\infty \exp(-su)f(u + T) \, du \\
&= \int_0^T \exp(-st)f(t) \, dt + \exp(-sT) \int_0^\infty \exp(-su)f(u) \, du \\
&= \int_0^T \exp(-st)f(t) \, dt + \exp(-sT)\mathcal{L}(f(t))(s).
\end{align*}
$$

This gives the result.
1.18 Models that lead to Systems of DE’s

Consider an electric circuit containing two closed loops as in Figure 10.

Here $R_1$ and $R_2$ are the resistances of the corresponding resistors, $q_1(t)$ and $q_2(t)$ the charges of the corresponding capacitors, $i_1(t)$, $i_2(t)$ and $i(t)$ are the currents in the corresponding parts of the circuit, $L$ is the inductance of the (only) inductor, and $E(t)$ is the electromotive force.

Using Kirchoff’s Current Law for the first junction, one gets $i_1 - i_2 - i = 0$, i.e. $i = i_1 - i_2$. The same equality follows from the second junction.

Next, applying Kirchoff’s Voltage Law to the left loop, it follows that

$$R_1 i_1(t) + L \frac{di_1}{dt}(t) + \frac{1}{C_1} q_1(t) = E(t).$$

Using $i = i_1 - i_2$ and $i_j(t) = q_j'(t)$ for $j = 1, 2$, we get

$$L[q_1''(t) - q_2''(t)] + R_1 q_1'(t) + \frac{1}{C_1} q_1(t) = E(t).$$

Similarly, it follows from Kirchoff’s Voltage Law applied to the right loop that

$$-L[q_1''(t) - q_2''(t)] + R_2 q_2'(t) + \frac{1}{C_2} q_2(t) = 0.$$ 

The equations (11) and (12) form a system of DE’s that determines $q_1(t)$ and $q_2(t)$ provided some initial conditions for these are given.

![Figure 10: Circuit with two loops](image.png)
1.19 Systems of (constant-coefficient) first order d.e.s

The last subsection of these Laplace transform notes dealt with an example involving a pair of second-order linear d.e.s. Any system of higher order equations can be written as a first order system, and any systematic treatment deals with first order systems.

In earlier parts of the course we considered, with $A$ an $n$ by $n$ matrix, (and $f(t)$ a given ‘forcing’ function) linear systems

$$\frac{dy}{dt} - Ay = f(t) .$$

We defined an initial-value problem for such a system as the problem of finding a function $y(t)$ satisfying the equation above and also satisfying $y(0)$ being equal to some prescribed vector (often denoted $y_0$).

While the Matlab Symbolic Toolbox `dsolve` function is perhaps the best way of treating the problem when a closed form exact solution is anticipated, it is worth noting too that Laplace transforms are eminently well-suited to solving the problem. If the Laplace transform of (the components of) $f$ can be found, we can find the Laplace transform of $y$. It may happen that the inversion of the Laplace transform is difficult (or even impossible in terms of nice functions), but it happens that there is further theory associated with Laplace transforms which extracts partial information (for example, – details not treated in ENVE3605 – behaviour at large time $t$ for $y(t)$ can be related to the behaviour at small $s$ of $Y(s) = \mathcal{L}(y)(s)$).

1.19.1 Homogeneous d.e.s

In the earlier part of the course we saw that the homogeneous problem ($f \equiv 0$) is solved using matrix exponentials:

$$f(t) \equiv 0, \quad \text{then } y(t) = \text{exponential}(At)y_0 .$$

Now consider taking the Laplace transform. With

$$Y(s) = \mathcal{L}(y)(s) ,$$

we have

$$\mathcal{L}\left(\frac{dy}{dt}\right)(s) = sy(s) - y_0 .$$
Hence, taking the Laplace transform of the system of d.e.s leads to

\[ sY(s) - AY(s) = y_0 , \]

or, equivalently,

\[ Y(s) = (sI - A)^{-1}y_0 . \]

On noting that

\[ \mathcal{L}({\text{exponential}}(At))(s) = (sI - A)^{-1} , \]

one can see that the two different approaches to solving the initial-value problem (the matrix exponential approach which we treated in the matrix algebra section of ENVE3605) and this Laplace transform approach are consistent.

If one multiplies the preceding matrix by the characteristic polynomial of \( A \), as a function of \( s \), we get a matrix all of whose entries are polynomials in \( s \).

Note the singular behaviour when \( s \) is an eigenvalue of \( A \).

### 1.19.2 Nonhomogeneous d.e.s

We now turn to the nonhomogeneous problem, \((N)\). Taking the Laplace transform (and using the initial-values in doing so) gives:

\[ sY(s) - AY(s) = F(s) + y_0 , \]

where \( F(s) = \mathcal{L}(f)(s) \). Thus

\[ Y(s) = (sI - A)^{-1} (F(s) + y_0) . \]

Again it is just a matter of inverting this and we will have the solution \( y(t) \) to the initial-value problem. One of the terms involves the product of the Laplace transform \( F(s) \) with the Laplace transform of the exponential matrix of \( At \). The Convolution Theorem enables this to be viewed as the convolution of the exponential matrix and of the forcing \( f(t) \), and this gives an alternative route through to the ‘Variation of Parameters’ formula we mentioned in the Linear Algebra notes on ‘Systems of D.E.s’.
1.20 Further topics

In these notes we have merely introduced you to some of the first results concerning Laplace Transforms. Here are a few that you may see in later engineering work:

- There is an inversion formula for Laplace Transforms. This, however, involves integrations along paths in the complex plane. It is frequently derived from the inversion formula for complex Fourier transforms.

- There are times when we can find the Laplace transform of our unknown but cannot do the inversion exactly. In situations like this there are various approaches to getting some information on the solution without having a formula for the solution.
2 Matlab’s Symbolic Toolbox for laplace transforms

2.1 Getting laplace transforms from your Matlab’s Symbolic Toolbox

Note that Matlab’s Symbolic Toolbox can compute the Laplace transform of (sufficiently simple) functions. For example to compute the Laplace transform of \( \cos(3t) \) you type in the following one-liner code.

```matlab
% Matlab Symbolic Toolbox
syms t s
laplace('cos(3*t)', t, s)
```

Here \( t \) is the variable in \( \cos(3 \times t) \) and \( s \) is the transform variable.

2.2 Computing the inverse Laplace transform

Matlab’s Symbolic Toolbox can help you with the (usually) difficult problem of inverting a Laplace transform (ie computing \( L^{-1} \)). As an example, let us consider the problem in which you are to solve the initial value problem:

\[
y'' - 2y' + 2y = \cos(t) , \quad y(0) = 1 , \quad y'(0) = 0 .
\]

If \( Y(s) = L(y(t)) \), you can compute (by hand!) that \( Y \) is given by

\[
Y(s) = \frac{s^2 + 1 + s - 2}{s^2 - 2s + 2}
\]

To find \( y(t) = L^{-1}(Y(s)) \) type the following one-liner commands in Matlab’s Symbolic Toolbox:

```matlab
% Matlab Symbolic Toolbox
syms s t Y
Y = (s/(s^2+1)+s-2)/(s^2-2*s+2);
ilaplace(Y,s,t)
```
2.3 Partial fractions

Given a rational function \( r \) as a ratio of polynomials in \( s \), the CAS underlying the Symbolic Toolbox can find partial fractions. Entering into Matlab

\[
\text{doc(symengine,’partfrac’)};
\]

produces the (MuPAD) documentation to find partial fractions.

Matlab allows people with a Maple licence to use Maple as the underpinning of the Symbolic Toolbox. Such users can access all of Maple’s functionality via a \texttt{maple} command. E.g. Maple’s

\[
\text{convert(r,’parfrac’,’s’)};
\]

finds the partial fraction expansion.

The \texttt{partfrac} capability does not appear to be available as a one-liner in Matlab’s Symbolic Toolbox. (For float numeric equivalents, see the \texttt{residue} function, and, as an example of its use, HDR Matlab Example 9.6.) Of course, using \texttt{solve} allows one to do partial fraction decompositions: see the Web pages for code for an example.

2.4 Solving an ODE in Matlab’s Symbolic Toolbox via Laplace transforms

You can use Matlab’s Symbolic Toolbox to solve the entire problem. For simplicity we shall rewrite the differential equation as

\[
y'' - 2y' + 2y - \cos(t) = 0
\]

Here is the Matlab’s Symbolic Toolbox code.

\[
\text{\% Matlab Symbolic Toolbox}
\text{ode= ’D(D(y))(t) -2*D(y)(t) + 2*y(t) -cos(t) = 0’}
\text{\% Next we take the Laplace transform of the equation}
\text{sym s t}
\text{ltode = laplace(ode,t,s)}
\text{\% Tidy the equation and use initial conditions}
\text{sym Y}
\]

36
neweqn = subs(ltode, ...
{'laplace(y(t),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 1, 0})
% Next we solve to obtain the Laplace transform of the solution:
ytrans = solve(neweqn, Y)
% Finally we can invert the Laplace transform to obtain the solution:
ysol = ilaplace(ytrans, s, t)

syms y
dsolve('D2y-2*Dy+2*y=cos(t)', 'y(0)=1', 'Dy(0)=0', 't')
simplify(ans-ysol) % gives 0 which is a check

2.5 The Heaviside function and Dirac measure

Your Matlab’s Symbolic Toolbox knows about the Heaviside of UnitStep function and about the Dirac Delta function. Two new ‘functions’ introduced in HDR Chpt 9 or equivalently Chapter 3 of ON. The unit step function $u_c(t)$ is given by $\text{Heaviside}(t-c)$. Thus you should be able to guess what the following one-liner commands will do:

% Matlab Symbolic Toolbox % numerical Matlab uses stepfun
syms t s
ezplot('stepfun(t,2)',[0 4])
% laplace('Heaviside(t-2)',t,s) % Maple version
laplace('heaviside(t-2)',t,s) % MuPAD version

Your Matlab’s Symbolic Toolbox also knows about the Dirac $\delta$-function. It isn’t really a function, of course. See what happens when you type the one-liners:

% Matlab Symbolic Toolbox
syms t s
% laplace('Dirac(t-1)',t,s) % Maple version
laplace('dirac(t-1)',t,s) % MuPAD version
3 Table of Laplace Transforms

Use this when you don’t have Matlab’s Symbolic Toolbox available.

\[ \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt \]

<table>
<thead>
<tr>
<th>SPECIFIC FUNCTIONS</th>
<th>GENERAL RULES</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{s} ) ( n \in \mathbb{Z}^+ )</td>
<td>( \frac{e^{-cs}}{s} ) ( F(s) )</td>
</tr>
<tr>
<td>( \frac{1}{s^p} ), ( p &gt; 0 )</td>
<td>( e^{-cs} )</td>
</tr>
<tr>
<td>( \frac{1}{s+a} ) ( n \in \mathbb{Z}^+ )</td>
<td>( F(s+a) )</td>
</tr>
<tr>
<td>( \frac{1}{s^2+\omega^2} ) ( \omega &gt; 0 )</td>
<td>( sF(s) - f(0) )</td>
</tr>
<tr>
<td>( \frac{1}{(s+a)^2+\omega^2} ) ( \omega &gt; 0 )</td>
<td>( s^2F(s) - sf(0) - f'(0) )</td>
</tr>
<tr>
<td>( \frac{1}{(s^2+\omega^2)^2} ) ( \omega &gt; 0 )</td>
<td>( F''(s) )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{s^2+\omega^2}} ) ( \omega &gt; 0 )</td>
<td>( \int_0^t f(\tau),d\tau )</td>
</tr>
<tr>
<td>( \frac{1}{s^m} ) ( m \geq 1 ) ( F(m) )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{m} ) ( F\left( \frac{s}{m} \right) )</td>
<td></td>
</tr>
<tr>
<td>( \frac{t\sin(\omega t)}{2\omega} ) ( J_0(\omega t) )</td>
<td></td>
</tr>
<tr>
<td>( \frac{e^{-at}}{(n-1)!} \Gamma(p) ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\cos(\omega t)}{\omega} ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\sin(\omega t)}{\omega^2} ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\tan(\omega t)}{2\omega} ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\sinh(\omega t)}{\omega^2} ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\cosh(\omega t)}{\omega^2} ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\sinh(\omega t)}{\omega} ) ( e^{-at} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{\cosh(\omega t)}{\omega} ) ( e^{-at} )</td>
<td></td>
</tr>
</tbody>
</table>

Higher derivatives

\[ \mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \]

Periodic functions of period \( T \), \( f(t+T) = f(t) \),

\[ \mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st}f(t)\,dt. \]

\[ \frac{1}{1 - e^{-Ts}} = 1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots. \]

The Convolution Theorem

\[ h(t) = (f \otimes g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \quad \text{then} \quad \mathcal{L}\{h\} = \mathcal{L}\{f\}\mathcal{L}\{g\}. \]