Assignment 2: This assignment counts for 10% of the assessment\(^1\) for 3P2 in 2005. Solutions are due by 4pm on Tuesday May 3, 2005.

1. Let \( Z \equiv \{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^3 = 0 \} \). Prove that \( Z \) is not a smooth manifold. What about \( Z_{\pm} \equiv \{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^3 = \pm 0.01 \} \)?

2. Give an example of a subset \( Z \) of some \( \mathbb{R}^n \), and \( x \in Z \) such that \( T_Z x \) is not a vector subspace of \( \mathbb{R}^n \). Prove that \( T_Z x \) is not a vector subspace of \( \mathbb{R}^n \).

3. Show that stereographic projection \( \phi_+: S^n \to \mathbb{R}^n \) from the north pole \((0,0,\ldots,1)^{\top} \in \mathbb{R}^{n+1}\) is a diffeomorphism, for any integer \( n \geq 1 \).

4. Construct an embedding \( f \) of the real projective plane \( \mathbb{R}P^2 \) in \( \mathbb{R}^4 \), and prove that \( f \) is an embedding, namely that \( f \) is continuous and has a continuous inverse defined on its image \( f(\mathbb{R}P^2) \subset \mathbb{R}^4 \).

5. Let \( S^3 \) be the unit sphere in \( \mathbb{R}^4 \), identified with the unit quaternions. For \( q \in S^3 \) and \( r \in \mathbb{R}^4 \) define

\[
\xi(q)r = q.r.q^{-1}
\]

where multiplication and inversion are quaternionic. Prove

(a) \( \xi(q) \in SO(3) \) for all \( q \in S^3 \)
(b) \( \xi(q) = \xi(p) \iff p = \pm q \) where \( p, q \in S^3 \)
(c) \( \xi : S^3 \to SO(3) \) is a group homomorphism with kernel \( \pm (1,0,0,0) \)
(d) \( \xi \) maps onto the whole of \( SO(3) \).

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\(^1\)Plagiarism is a serious offence under faculty rules (either copying or permitting it).
1. Prove the second straightening theorem:

Given $C^\infty f : D \to \mathbb{R}^m$, where $D$ is open in $\mathbb{R}^n$ and $n < m$, suppose $df_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ has rank $n$. Then there is an open subset $U$ of $\mathbb{R}^m$ containing $y_0 \equiv f(x_0)$ and a $C^\infty$ diffeomorphism $F$ from $U$ onto an open subset $V$ of $\mathbb{R}^m$ with the property that

$$F \circ f(x) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) \in \mathbb{R}^m.$$