Assignment 3: This assignment counts for 10% of the assessment\textsuperscript{1} for 3P2 in 2005. Solutions are due by 4pm on Thursday June 2, 2005, but you may hand your assignment in earlier if you wish and I will try to mark it as soon as possible.

1. A Lie group is a smooth manifold $G$ whose multiplication and inversion

$$G \times G \to G \quad \text{and} \quad G \to G$$

are smooth maps.

(a) Show that $S^3 \cong U$ is a Lie group with respect to quaternionic multiplication of unit quaternions.

(b) An $n \times n$ matrix $A$ of complex numbers is said to be unitary when $AA^\text{T}$ is the identity matrix (here $\bar{\cdot}$ means complex conjugate, and $^\text{T}$ means matrix transpose. Prove that the space $U_n$ of all unitary $n \times n$ matrices is a Lie group. What is its dimension? What is the tangent space $T(U_n)_1$ at the identity $1$?

2. Let $G$ be a Lie group of dimension $n$. Show that there are smooth vector fields $X_1, X_2, \ldots, X_n$ on $G$ such that $\{X_1(x), X_2(x), \ldots, X_n(x)\}$ is a basis of $TG_x$ for every $x \in G$.

3. Let $G$ be a Lie group. Left multiplication by $g \in G$ is the smooth function $L(g) : G \to G$ given by $L(g)h = gh$. A vector field $X$ on $G$ is said to be left-invariant when $\bar{X}(gh) = dL(g)_h(\bar{X}(h))$ for all $g, h \in G$. Prove that if $X, Y$ are left-invariant so is their Lie bracket $[X, Y]$.

4. Define vector fields $X, Y$ on $\mathbb{R}^2$ by

$$\bar{X}(x, y) = (xy, x) \quad \text{and} \quad \bar{Y}(x, y) = (x + e^y, y).$$

Calculate the Lie bracket $[X, Y]$.

\textsuperscript{1}Plagiarism is a serious offence under faculty rules (either copying or permitting it).