Assignment 2: This assignment counts for 10% of the assessment for 3P2 in 2005. Solutions were due by 4pm on Tuesday May 3, 2005.

1. Let \( Z \equiv \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^3 = 0\} \). Prove that \( Z \) is not a smooth manifold. What about \( Z_\pm \equiv \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^3 = \pm 0.01\} \)?

**Solution:** Let \( \omega \equiv (\omega_1, \omega_2) : (-\epsilon, \epsilon) \to Z \) be a smooth curve in \( Z \) satisfying \( \omega(0) = (0, 0) \). Then \( \omega_1(t)^2 + \omega_2(t)^3 = 0 \) for all \( t \in (-\epsilon, \epsilon) \). Differentiating,

\[
2\dot{\omega}_1(t)\omega_1(t) + 3\dot{\omega}_2(t)\omega_2(t)^2 = 0
\]

Differentiating again,

\[
2\ddot{\omega}_1(t)\omega_1(t) + 2\dot{\omega}_1(t)^2 + 3\ddot{\omega}_2(t)\omega_2(t)^2 + 6\dot{\omega}_2(t)^2\omega_2(t) = 0
\]

and, setting \( t = 0 \), we find \( \dot{\omega}_1(0) = 0 \). Differentiating again, and then setting \( t = 0 \),

\[
0 + 0 + 0 + 0 + 0 + 6\ddot{\omega}(0)^3 = 0
\]

namely \( \ddot{\omega}(0) = 0 \). So \( TZ_{(0,0)} = \{(0,0)\} \). Define \( \mu : \mathbb{R} \to Z \) by \( \mu(t) = (t^3, -t^2) \). Then \( \dot{\mu}(1) = (3, -2) \neq (0,0) \) and so \( TZ_{(1,-1)} \) is at least (exactly) 1-dimensional. So \( Z \) is neither a 0-manifold nor a 1-manifold: it is not a manifold at all.

Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(x_1, x_2) = x_1^2 + 2x_2^3 \). Then

\[
df_{(x_1,x_2)}(v_1, v_2) = 2x_1 v_1 + 3x_2^2 v_2
\]

and \( df_{(x_1,x_2)} \) is therefore nontrivial except when \( x_1 = x_2 = 0 \). It follows that \( 0 = f(0,0) \) is the only critical value of \( f \). So \( Z_\pm = f^{-1}(0.01) \) are smooth \( 2-1 = 1 \)-manifolds.

2. Give an example of a subset \( Z \) of some \( \mathbb{R}^n \), and \( x \in Z \) such that \( TZ_x \) is not a vector subspace of \( \mathbb{R}^n \). Prove that \( TZ_x \) is not a vector subspace of \( \mathbb{R}^n \).

**Solution:** Set \( Z = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^2 \), namely

\[
Z = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}
\]

If \( \omega : (-\epsilon, \epsilon) \to Z \) is a smooth curve in \( Z \) satisfying \( \omega(0) = (0,0) \) then \( \omega_1(t)\omega_2(t) = 0 \) for all \( t \). Differentiating,

\[
\dot{\omega}_1(t)\omega_2(t) + \omega_1(t)\dot{\omega}_2(t) = 0
\]

Differentiating again,

\[
\ddot{\omega}_1(t)\omega_2(t) + 2\dot{\omega}_1(t)\dot{\omega}_2(t) + \omega_1(t)\ddot{\omega}_2(t) = 0
\]

and setting \( t = 0 \) we find \( \dot{\omega}_1(0)\omega_2(0) = 0 \). So \( TZ_{(0,0)} \subseteq Z \). Considering the curves \( t \mapsto (t,0) \) and \( t \mapsto (0,t) \) we see \( TZ_{(0,0)} = Z \). However \( Z \) is not closed under vector addition.
3. Show that stereographic projection \( \phi_+ : S^n \to \mathbb{R}^n \) from the north pole \((0,0,\ldots,1) \in \mathbb{R}^{n+1}\) is a diffeomorphism, for any integer \( n \geq 1 \).

**Solution:** We find
\[
\phi_+(x) = \frac{(x_1, x_2, \ldots, x_n)}{1 - x_{n+1}}
\]
where \( x \in S^n - \{(0,0,\ldots,1)\} \), and
\[
\phi_+^{-1}(y) = \left(\frac{2y, \|y\|^2 - 1}{\|y\|^2 + 1}\right)
\]
where \( y \in \mathbb{R}^n \). \( \phi_+^{-1} \) is smooth because it is a rational function whose denominator never vanishes. \( \phi_+ \) is smooth because it extends to the open subset \( \mathbb{R}^{n+1} - \mathbb{R}^n \times \{1\} \) of \( \mathbb{R}^n \) (the pre-image by the (linear) projection to the \( n+1 \)th coordinate of the open subset \( \mathbb{R} - \{1\} \) of \( \mathbb{R} \)).

4. Construct an embedding \( f \) of the real projective plane \( \mathbb{R}P^2 \) in \( \mathbb{R}^4 \), and prove that \( f \) is an embedding, namely that \( f \) is continuous and has a continuous inverse defined on its image \( f(\mathbb{R}P^2) \subset \mathbb{R}^4 \).

**Solution:** Let \( \pi : S^2 \to \mathbb{R}P^2 \) be the canonical projection of the equivalence relation given by \( x \cong -x \) for \( x \in S^2 \). Defining \( g : S^2 \to \mathbb{R}^4 \) by
\[
g(x) = (x^2_1, x^2_2 + x_1x_3, x_1x_2, x_2x_3)
\]
we have \( g(x) = g(-x) \). So there is a unique map \( f : \mathbb{R}P^2 \to \mathbb{R}^4 \) satisfying \( g = f \circ \pi \). Since \( g \) is continuous (polynomial), \( f \) is continuous (defining property of the quotient topology).

Since \( S^2 \) is closed (preimage of \( \{1\} \) under \( x \mapsto \|x\|^2 \)) and bounded (definition of boundedness), \( S^2 \) is compact (Heine-Borel Theorem). Since \( \pi \) is continuous, \( \pi(S^2) = \mathbb{R}P^2 \) is also compact. Since \( \mathbb{R}^4 \) is a metric space so is \( f(\mathbb{R}P^2) \) with the subspace topology, and therefore \( f(\mathbb{R}P^2) \) is Hausdorff. So it suffices (Lemma in lectures) to prove that \( f \) is one-to-one, namely that \( g(x) = g(y) \Rightarrow x = \pm y \).

Let \( g(x) = g(y) \), and suppose first that \( x_1 = 0 \). Then \( y_1 = 0 \) and \( x^2_2 = y^2_2 \). If \( x_2 = 0 \) then \( y_2 = 0 \) and \( x = (0,0,\pm1) = y \). If \( x_2 \neq 0 \) then at least \( x_2 = \sigma y_2 \) where \( \sigma = \pm 1 \). Since \( x_2x_3 = y_2y_3 \) and \( x_2 \neq 0 \neq y_2 \) we then have \( x_3 = \sigma y_3 \) and \( x = (0,x_2,x_3) \), \( y = (0,\sigma x_2,\sigma x_3) = \sigma x \).

If, alternatively, \( x_1 \neq 0 \) then \( x_1 = \sigma y_1 \) where \( \sigma = \pm 1 \). Since \( x_1x_2 = y_1y_2 \) we then have \( x_2 = \sigma y_2 \). Since \( x_2x_3 = y_2y_3 \) we have \( x_3 = \sigma y_3 \). So \( x = \sigma y \).

5. Let \( S^3 \) be the unit sphere in \( \mathbb{R}^4 \), identified with the unit quaternions. For \( q \in S^3 \) and \( r \in \mathbb{R}^4 \) define
\[
\xi(q)r = q.r.q^{-1}
\]
where multiplication and inversion are quaternionic. Prove
(a) \( \xi(q) \in SO(3) \) for all \( q \in S^3 \)

Solution: Evidently \( \|q\|^2 = q\bar{q} \), for any quaternion \( q \). So

\[
\|qr\|^2 = qr\bar{q} = q\|r\|^2\bar{q} = q\bar{q}\|r\|^2 = \|q\|^2\|r\|^2
\]

and \( \|qr\| = \|q\|\|r\| \). So

\[
\|\xi(q)\| = \|qr\|\|\bar{q}\| = \|q\|\|r\|\|r\| = \|r\|
\]

since \( q \) is a unit quaternion. Now \( \xi(q) \) is linear in \( r \) and preserves lengths. So \( \xi(q) \in O(3) \). To prove \( \xi(q) \in SO(3) \) it suffices to show \( \det \xi(q) = 1 \).

But \( \det \circ \xi : S^3 \to \{\pm 1\} \subset \mathbb{R} \) is continuous (restriction of a polynomial map on \( \mathbb{R}^4 \)) and \( S^3 \) is path-connected (use great-circle arcs). So \( \det \circ \xi(S^3) \) is path-connected. The nonempty path-connected subsets of \( \{\pm 1\} \) are \( \{1\} \) and \( \{-1\} \) (exclude \( \{\pm 1\} \) by a continuity-supremum argument), and so \( \det \circ \xi \) is constant. Since \( \det \circ \xi(1) = 1 \), \( \det \circ \xi \) is identically 1 and \( \xi(q) \in SO(3) \) for all \( q \in S^3 \).

(b) \( \xi(q) = \xi(p) \iff p = \pm q \) where \( p, q \in S^3 \)

Solution: Because \( \xi(q) \) is homogeneous and quadratic in the components of \( q \),

\( \xi(q) = \xi(-q) \). Conversely, if \( \xi(q) = \xi(p) \) then \( \xi(s) = 1_{\mathbb{R}^3} \) where \( s \equiv qp^{-1} = q\bar{p} \), because \( \xi \) is a homomorphism (part c). So it suffices to show that the kernel of \( \xi \) is \( \{\pm 1\} \) (part c).

(c) \( \xi : S^3 \to SO(3) \) is a group homomorphism with kernel \( \pm(1, 0, 0, 0) \)

Solution: For any \( u, v \in S^3 \) and \( r \in \mathbb{R}^3 \), \( \xi(uv)r = uv\bar{v}\bar{u} = \xi(u)(v\bar{v}) = \xi(u)\circ \xi(v)(r) \). So \( \xi(uv) = \xi(u)\circ \xi(v) \), namely \( \xi \) is a homomorphism.

If \( \xi(s) = 1_{\mathbb{R}^3} \) for any \( s \in S^3 \) then we have \( sr\bar{s} = r \) for all pure imaginary quaternions \( r \). Let \( s = s_R + s_I \) where \( s_R \) and \( s_I \) are real and imaginary respectively. Let \( r \) be a unit vector orthogonal to \( s_I \). Since \( r \) anticommutes with \( s_I \)

\[
sr\bar{s} = (s_R + s_I)r(s_R - s_I) = (s_R^2 + 2s_Rs_I + s_I^2)r = (1 + 2s_Rs_I)r
\]

Because \( s \) is a unit quaternion the right hand side \( (r) \) is equal to \( (s_R^2 - s_I^2)r \) and so

\[
s_R^2 + 2s_Rs_I + s_I^2 = s_R^2 - s_I^2
\]

Then \( 2s_Rs_I = 2s_I^2 \) and if \( s_I \neq 0 \) we have \( s_I = s_R \) (impossible). So \( s_I = 0 \) and \( s \) is real. Then \( s = \pm 1 \), since \( s \) is a unit quaternion.

(d) \( \xi \) maps onto the whole of \( SO(3) \).

Solution: The characteristic polynomial \( p \) of any \( A \in SO(3) \) is cubic and has real coefficients. Its roots are the eigenvalues of \( A \) (complex numbers of length 1) and are either real or occur in complex conjugate pairs, since \( p \) has real coefficients.

If \( p \) has a complex root \( \lambda, \bar{\lambda} \) is another root, and the remaining root must therefore be \( \pm 1 \). Since \( \det A \) is the product of the eigenvalues, and \( \lambda\bar{\lambda} = 1 \) the remaining eigenvalue is 1.
If all the roots of $p$ are real they must be $\pm 1$, and they cannot all be $-1$ since then $\det A = -1$. So again $1$ is an eigenvalue of $A$.

So in any case, $A$ fixes some unit vector $w$ in $\mathbb{R}^3$, namely $A$ is a rotation in the plane $P$ orthogonal to $w$.

Fix unit vectors $u, v \in P$ so that $\{u, v, w\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. For $\theta \in \mathbb{R}$ set $q = \cos \theta + w \sin \theta$. Then

$$\xi(q)u = (\cos \theta + w \sin \theta)u(\cos \theta - w \sin \theta) = u(\cos^2 \theta - \sin^2 \theta) + 2v \sin \theta \cos \theta$$

since $u, w$ anticommute and $wu = v$. Similarly

$$\xi(q)(v) = (\cos \theta + w \sin \theta)v(\cos \theta - w \sin \theta) = -2u \sin \theta \cos \theta + v(\cos^2 \theta - \sin^2 \theta)$$

So $\xi(q)$ is rotation in the plane $P$ from $u$ towards $v$ by an angle $-2\theta$. Choosing $\theta$ appropriately, $\xi(q) = A$. 

1. Prove the second straightening theorem:

Given $C^\infty f : D \to \mathbb{R}^m$, where $D$ is open in $\mathbb{R}^n$ and $n < m$, suppose $df_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ has rank $n$. Then there is an open subset $U$ of $\mathbb{R}^m$ containing $y_0 \equiv f(x_0)$ and a $C^\infty$ diffeomorphism $F$ from $U$ onto an open subset $V$ of $\mathbb{R}^m$ with the property that

$$F \circ f(x) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) \in \mathbb{R}^m.$$