Concepts in Analysis

Lyle Noakes

April 22, 1999
## Contents

Preface ........................................................................... 5

1 Trigonometric Approximations ...................................... 9
   1.1 Two Problems .................................................. 9
   1.2 Relationships Between Problems 1 and 2 .................. 11
   1.3 Solving Problem 2 in a Special Case ....................... 13
   1.4 Solving Problems 1 and 2 ..................................... 14
   1.5 Even and Odd Functions .................................... 18
   1.6 Examples ....................................................... 21

2 Vector Spaces of Functions .......................................... 27
   2.1 Vector Spaces .................................................. 28
   2.2 Linear Combinations ......................................... 32
   2.3 Linear Transformations ...................................... 36
   2.4 Linear Isomorphisms ......................................... 39
   2.5 Matrices ........................................................ 41
   2.6 Norms .......................................................... 44
   2.7 Inner Products ................................................ 46
   2.8 Inner Products and Norms .................................. 47
   2.9 Orthonormal Sets .............................................. 49
   2.10 Parseval’s Identity ........................................... 51

3 Convergence of Trigonometric Series .............................. 55
   3.1 Infinite Series .................................................. 55
      3.1.1 Power Series ............................................. 58
      3.1.2 Trigonometric Series .................................. 60
## CONTENTS

3.2 Convergence of Sequences ..................................... 60  
3.3 Sequences and Series of Vectors .......................... 63  
3.4 Sequences and Series of Functions ..................... 66  
3.5 The Uniform Norm ........................................... 67  
3.6 The Riemann Integral ....................................... 72  
3.7 Square-Integrable Functions ................................. 74  
3.8 The Lebesgue Integral ...................................... 74  
3.9 Pointwise Convergence of Fourier Series ............... 76  
3.10 The Gibbs Phenomenon .................................... 80  
3.11 Parseval’s Identity ......................................... 82  
3.12 Convolutions of Functions ................................ 83  
3.13 Convolutions of Sequences ................................. 87  

4 Partial Derivatives and Applications ....................... 89  
4.1 Heat Flow and Laplacians ................................ 89  
4.2 Vibrations and Laplacians ................................ 90  
4.3 Gradient and Curl ........................................... 92  
4.4 Stokes’ Theorem and Divergence .......................... 94  
4.5 Differentiation Following Fluid Motion .................. 96  
4.6 Static Charges in Free Space ............................... 98  
4.7 Electrostatics ............................................... 101  
4.8 Moving Charges ............................................. 104  
4.9 Maxwell’s Equations ...................................... 107  
4.10 The Wave Equation ......................................... 109  
4.11 The Telegraphist’s Equation ............................... 113  

5 Partial Differential Equations ................................. 115  
5.1 The Heat Equation .......................................... 117  
5.2 The Wave equation ......................................... 121  

A Different Conventions ......................................... 127  
A.1 Fourier Sums and Series .................................. 127
Preface

About these Notes

Hi, welcome to this website! These notes have been used as background for the second-year course

2CA1 Concepts in Analysis

in Semester 1, 1999 in the Department of Mathematics & Statistics at the University of Western Australia.

The Department of M&S has not approved these notes. Nor are the notes in any way essential for the 2CA1 course (the lecture notes define the course).

People who read these notes are expected to know something about

- real and complex numbers,
- limits, continuity and differentiability of functions,
- differential equations,
- functions of several variables, and partial derivatives,
- sequences and series,
- vector spaces, linear transformations, and matrices.

These are things you might reasonably expect to know after a first course in mathematics at an Australian university.

In order to understand some of the applications, it wouldn’t hurt to know a little bit of mathematical physics, especially applications using partial differential equations.
These notes start out talking about approximations of functions by linear combinations of trigonometric functions. There’s a related problem of approximating functions by linear combinations of exponential functions. Some simple facts are proved, and these are used to derive some interesting relationships between $\pi$ and infinite series of rational numbers.

At this stage, still in Chapter 1, the stuff on infinite series is done intuitively. It means that you can wait until Chapter 3 before brushing up on sequences and series of real numbers.

Meanwhile, in Chapter 2, there’s some revision of linear algebra, with a lot of emphasis on vector spaces of functions. The first vector space that everyone meets is $\mathbb{R}^m$ then $\mathbb{C}^m$ where $m \geq 1$. In Chapter 2 we move on to talk about vector spaces that can’t easily be visualised, because each vector is actually a function. Nevertheless it turns out that geometric concepts which are useful in $\mathbb{R}^m$ can also be applied in these function spaces.

Apart from the notion of an abstract vector space itself, one of the most useful concepts that we generalise from $\mathbb{R}^n$ is that of an inner product. The notion of a norm is also defined and used, in the same way as in $\mathbb{R}^n$, to define distances between vectors.

It then turns out that simple results in linear algebra can be used to rewrite and very quickly prove some of the things we discovered in Chapter 1. Completely new things about approximations by trigonometric sums are also proved, especially Parseval’s Identity. These things are mainly expressed in terms of approximations by trigonometric sums, and up to this point it is possible to get along without using infinite sequences or series.

In Chapter 3 we finally introduce sequences and series in general normed vector spaces. Convergence of such things is talked about, and a few simple facts are proved. The main application is to rewrite some of the results of Chapters 1 and 2 in terms of infinite series of trigonometric and exponential functions. Once the language is understood the results take on a simpler and cleaner appearance.

Up to this point the main application of the results in the notes is to derive some interesting and curious expressions for $\pi$ in terms of infinite series. In Chapter 4, however, we start to use Fourier series to solve some partial differential equations of mathematical physics. This is what Fourier series were invented for, and it’s still the most important application. Partial differential equations are used to model a wide range of natural phenomena, including the flow of heat, the vibration of membranes, electromagnetic phenomena, and so on.
Another less classical application of the techniques in this course is to data smoothing and data compression, particularly image compression.

Whether you’re an engineering, mathematics, computer science or physics student, you’ll probably find uses for the materials in these notes.

In any case, happy reading!
Chapter 1

Trigonometric Approximations

This book is about some concepts in analysis that are useful for studying equations of mathematical physics. For this reason the concepts have applications in engineering, as well as in physics and mathematics. It will take a while before the connection with the important applications can be made, but I promise to make that connection. In the meantime we’ll be look at some interesting and nontrivial applications; mainly identities relating $\pi$ and infinite series.

Later on we’ll see some practical applications of trigonometric sums and series.

1.1 Two Problems

Let $y : \mathbb{R} \to \mathbb{R}$ be a real-valued function of a single real variable $x$. Consider the following problem.

**Problem 1:** Given $m \geq 1$ how can we choose constants

$$a_0, a_1, a_2, \ldots a_m \text{ and } b_1, b_2, \ldots b_m \in \mathbb{R}$$

so that $y(x)$ is best approximated by the sum

$$FS(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + \ldots a_m \cos(mx) + b_1 \sin x + b_2 \sin(2x) + \ldots b_m \sin(mx)$$

(1.1) 

The function $FS$ is called a *Fourier sum*, and we point to the slightly peculiar treatment of $a_0$. Division by 2 in (1.1) has the effect of tidying up formulas later on. Next notice that $FS$ is periodic of period $2\pi$, namely

$$FS(x + 2\pi) = FS(x)$$

for all $x \in \mathbb{R}$.
So unless the function $y$ is also periodic it doesn’t make much sense to approximate it by $FS$, unless of course $x$ is chosen to lie in some suitable subinterval of $\mathbb{R}$.

Accordingly, from now on we restrict the domain of $y$ to be the half-open interval $[-\pi, \pi)$, and suppose that for Problem 1 $x$ lies in this domain.

Still, we have a bit of a difficulty. It’s not at all clear why anybody would want to do this kind of approximation. Trigonometric sums are not particularly elementary functions. Polynomials of the form

$$P(x) = p_0 + p_1 x + p_2 x^2 + \ldots p_m x^m$$

are generally much easier to work out, and there are sound reasons for wanting to approximate complicated functions by polynomials. That’s what Taylor series are for. Later we’ll give excellent reasons for wanting to approximate a function $y$ by a Fourier sum $FS$. Right now we’re not in a position to say.

What we know about Taylor series isn’t much use for computing Fourier sums. It turns out that the best approach is to look at a generalisation of Problem 1. To state the generalisation let’s permit $y$ to take on (possibly) complex values namely

$$y : [-\pi, \pi) \rightarrow \mathbb{C}$$

Then we may state:

**Problem 2:** Given $m \geq 1$ how can we choose constants

$$c_{-m}, c_{-m+1}, \ldots c_{-1}, c_0, c_1, \ldots c_{m-1}, c_m \in \mathbb{C}$$

so that $y(x)$ is best approximated by an exponential sum of the form

$$ES(x) = \sum_{n=-m}^{m} c_n e^{inx}$$

where $x \in [-\pi, \pi)$.

Here $e \approx 2.718$ is the number satisfying $\ln e = 1$ where

$$\ln : (0, \infty) \rightarrow \mathbb{R}$$

is the natural logarithm function. By $i$ we mean, as usual, a complex number satisfying

$$i^2 = -1$$

In neither Problem 1 nor Problem 2 have we come out and said exactly what is meant by best approximated as $x$ varies over the half-open interval $[-\pi, \pi)$. This is something we don’t intend to define in this Chapter. In Chapter 2, however, we’ll be able to be a bit more helpful. In the meantime we’re just going to look for approximations that are sometimes good.
1.2 Relationships Between Problems 1 and 2

In Section 1.1 Problem 2 is claimed to be a generalisation of Problem 1. Let’s see now why this is so.

The first thing to note is the well-known identity expressing exponentials of imaginary numbers in terms of cos and sin.

\[ e^{ix} = \cos x + i \sin x \quad (1.3) \]

Replacing \( x \) by \( nx \) on both sides of (1.3) we obtain

\[ e^{nx} = \cos nx + i \sin nx \quad (1.4) \]

Substituting this into (1.2), \( ES(x) \) takes on the form (1.1)

\[
\frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + \ldots + a_m \cos mx + b_1 \sin x + b_2 \sin(2x) + \ldots + b_m \sin(mx)
\]

where (after a quick calculation) \( a_0 = 2c_0 \) and

\[
a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad (1.5)
\]

for \( 1 \leq n \leq m \). Notice that, unless we are very lucky with the \( c_n \), the constants \( a_0, a_1, \ldots, a_m \) and \( b_1, \ldots, b_m \) will not always be real numbers. They will, however, be perfectly good complex numbers.

Now let’s turn this around and try to write a trigonometric sum \( FS(x) \) in the form \( ES(x) \) for some choices of \( c_n \in \mathbb{C} \) where \( -m \leq n \leq m \). The \( c_n \) would have to satisfy the equations (1.5), and solving for \( c_n \) yields

\[
c_0 = \frac{a_0}{2}
\]

and

\[
c_n = \frac{a_n - ib_n}{2} \quad (1.6)
\]

and

\[
c_{-n} = \frac{a_n + ib_n}{2} \quad (1.7)
\]

for \( 1 \leq n \leq m \).

From Exercise 1.2.2 below it follows that, if we know how to solve Problem 2, then we can solve Problem 1.
Exercises

Exercise 1.2.1. Solve (1.3) to show that
\[ \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \]

Exercise 1.2.2. Prove that \( ES(x) \) can be written in the form \( FS(x) \) where
\[ a_0, a_1, a_2, \ldots a_m \]
and
\[ b_1, b_2, \ldots b_m \]
are real numbers if and only if the complex coefficients \( c_n \) satisfy
\[ \bar{c}_n = c_{-n} \text{ for all } -m \leq n \leq m \]
As usual \( \bar{c} \) means the complex conjugate of a complex number \( c \).

Solutions

Solution 1.2.1. \hfill \square

Solution 1.2.2. Hint: Replace \( n \) by \( -n \) in (1.6) and (1.7). \hfill \square
1.3 Solving Problem 2 in a Special Case

In order to avoid the need to say precisely what’s meant in Problem 2 by “best approximated” let’s focus on cases where it’s very clear what the best approximation is. For the remainder of this Section we restrict attention to the case where $y(x)$ is exactly of the form

$$ES(x) = \sum_{n=m}^{m} e^{nix}$$

Solving Problem 2 for such a function amounts to recovering the constants $c_n \in \mathbb{C}$ where $-m \leq n \leq m$

from values of $y(x) = \sum_{n=-m}^{m} e^{nix}$

There’s a particularly neat result that allows us to do this. In fact this result is the main reason for generalising from Fourier to exponential sums. It’s called a lemma. A lemma is a mathematical result that might not appear too impressive in itself, but turns out to be useful in proving more substantial results like propositions or even theorems. This lemma says something interesting about integrals of exponential powers.

**Lemma 1.3.1.** For $p, q \in \mathbb{Z}$

$$\int_{-\pi}^{\pi} e^{pix} e^{-qix} dx = 2\pi \delta_p^q$$

where $\delta_p^q$ is the Kronecker delta given by

$$\delta_p^q = 0 \text{ for } p \neq q, \text{ and } \delta_p^p = 1$$

**Proof:**

When $p = q$ the integrand on the left is identically 1, and so the value of the integral is $2\pi$. This proves the lemma in the case when $p = q$.

For $p \neq q$ we can rewrite the integral in the form

$$\int_{-\pi}^{\pi} e^{(p-q)ix} dx = \frac{1}{(p-q)i}(e^{(p-q)\pi i} - e^{-(p-q)\pi i}) =$$

$$\frac{1}{(p-q)i}(\cos((p-q)\pi) - \cos(-(p-q)\pi)) =$$

$$\frac{1}{(p-q)i}((-1)^{p-q} - (-1)^{p-q}) = 0 = 2\pi \delta_p^q$$

This proves the lemma. \qed
The next result is not substantial enough to be called a theorem, but it solves the problem of calculating the \( c_n \), at least when \( y(x) \) has exactly the form \( ES(x) \).

**Proposition 1.3.1.** When \( y(x) \) has exactly the form \( ES(x) \) for all values of the variable \( x \in [-\pi, \pi) \), the coefficients \( c_n \) in \( ES(x) \) can be calculated as the integrals

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-nix} \, dx
\]

for \( n \in \mathbb{Z} \).

**Proof:**
Substituting for \( y(x) = ES(x) \) in the expression on the right hand side, we obtain

\[
\frac{1}{2\pi} \sum_{p=-m}^{m} c_p \int_{-\pi}^{\pi} e^{p ix} e^{-nix} \, dx
\]

where \( p \) is used as the summation variable, to avoid confusion with \( n \). Using Lemma 1.3.1 this expression becomes

\[
\sum_{p=-m}^{m} c_p \delta_n^p = c_n
\]

The proposition is proved. \( \square \)

### 1.4 Solving Problems 1 and 2

Proposition 1.3.1 tells us how to approximate \( y(x) \) by \( ES(x) \) in the special case where \( y \) is exactly of this form. For more general functions \( y \), so long as the integrals exist, we can apply the formula

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-nix} \, dx \quad (1.8)
\]

in Proposition 1.3.1 to calculate the coefficients \( c_n \). What we will get in such a case is some kind of approximation to \( y \) rather than \( y \) itself. The integrals exist (for example) when \( y \) is piecewise continuous with at most finite jump discontinuities, and the approximation may be rather good, depending on the value of \( m \).

From (1.8) we can now get formulae for the coefficients \( a_0, a_1, a_2, \ldots a_m \) and \( b_1, b_2, \ldots b_m \) in Problem 1.
Proposition 1.4.1. When \( y : [-\pi, \pi) \to \mathbb{R} \) has exactly the form \( FS(x) \) for all values of the variable \( x \in [-\pi, \pi) \), the coefficients \( a_n, b_n \) in \( FS(x) \) can be calculated as the integrals

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx
\]

where \( n \geq 0 \) and \( n \geq 1 \) respectively.

For a more general function \( y : [-\pi, \pi) \to \mathbb{R} \) we can apply the same formulas as in Proposition 1.4.1

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx \quad (1.9)
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx \quad (1.10)
\]

to approximate \( y \) by the Fourier sum \( FS \). Notice that the formula for \( a_n \) does not change when \( n = 0 \). This is the reason for the 2 on the bottom line of the first term in \( FS(x) \).

Whereas exponential sums are neater for proving things, Fourier sums are more convenient for doing calculations.

Example 1.4.1. Let \( y(x) = e^x \) for \( x \in [-\pi, \pi) \). Then

\[
\pi a_n = \int_{-\pi}^{\pi} e^x \cos(nx) dx
\]

which, on integration by parts, becomes

\[
(-1)^n(e^\pi - e^{-\pi}) + n \int_{-\pi}^{\pi} e^x \sin(nx) dx
\]

and then

\[
(-1)^n(e^\pi - e^{-\pi}) - n^2 \int_{-\pi}^{\pi} e^x \cos(nx) dx = (-1)^n(e^\pi - e^{-\pi}) - n^2 \pi a_n
\]
It follows that

$$\pi a_n = (-1)^n \frac{e^\pi - e^{-\pi}}{1 + n^2}$$

Similarly

$$\pi b_n = \int_{-\pi}^{\pi} e^x \sin(nx) dx =$$

$$-n \int_{-\pi}^{\pi} e^x \cos(nx) dx = -n(-1)^n(e^\pi - e^{-\pi}) - n^2 \int_{-\pi}^{\pi} e^x \sin(nx) dx =$$

$$-n(-1)^n(e^\pi - e^{-\pi}) - n^2 \pi b_n$$

Consequently

$$\pi b_n = -n(-1)^n \frac{e^\pi - e^{-\pi}}{1 + n^2}$$

This gives (in the limit as $m \to \infty$)

$$\frac{\pi e^x}{e^\pi - e^{-\pi}} = \frac{1}{2} - \frac{1}{2} \cos x + \frac{1}{5} \cos(2x) - \frac{1}{10} \cos(3x) + \frac{1}{17} \cos(4x) - \ldots +$$

$$-\frac{1}{2} \sin x + \frac{2}{5} \sin(2x) - \frac{3}{10} \sin(3x) + \frac{4}{17} \sin(4x) - \ldots$$

Setting $x = 0$ gives

$$\frac{\pi}{e^\pi - e^{-\pi}} = \frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \frac{1}{26} + \ldots$$
Taking the first 15 terms on the right gives

$$0.1336227192$$

whereas the left hand side evaluates as

$$0.1360145275$$

Now let’s suppose that

$$\frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \frac{1}{26} + \ldots$$

has been worked out, at least to some approximation, and call its value $\xi$. Then

$$e^\pi - e^{-\pi} = \frac{\pi}{\xi}$$

Write $t = e^\pi$. Then $t > 0$ and we have a quadratic equation for $t$. Solving we get

$$t^2 - \frac{\pi}{\xi} t - 1 = 0$$

for $t$. Solving we get

$$t = \frac{\frac{\pi}{\xi} \pm \sqrt{\frac{\pi^2}{\xi^2} + 4}}{2}$$

and the positive solution is

$$e^\pi = \frac{\pi + \sqrt{\pi^2 + 4\xi^2}}{2\xi}$$

This is an interesting identity relating $e$, $\pi$ and the infinite sum $\xi$. Notice for instance that

$$\frac{1}{10} < \xi < \frac{27}{170}$$

It follows then that

$$\frac{\pi + \sqrt{\pi^2 + \frac{1}{25}}}{54} < e^\pi < 5\pi + 5\sqrt{\pi^2 + \left(\frac{27}{85}\right)^2}$$

□


**Exercises**

**Exercise 1.4.1.** Prove Proposition 1.4.1. 

*Hint:* Use (1.5) to write the $a_n, b_n$ in terms of the $c_n$. Then use Proposition 1.3.1 to express the $c_n$ in terms of integrals involving $y$. Finally, use (1.2.1) to replace exponential powers in the integral by expressions in $\cos(nx)$ and $\sin(nx)$.

*Solution 1.4.1.***

**1.5 Even and Odd Functions**

A function $y$ is said to be an *even function* when 

\[ y(-x) = y(x) \text{ for all values of } x \]

Constant functions are even, but there are many other examples.

**Example 1.5.1.** Let $P(x)$ be the polynomial 

\[ p_0 + p_1 x + \ldots + p_m x^m \]

If the coefficients $p_1, p_3, p_5, \ldots$ of odd powers of $x$ in $P(x)$ are all 0, then reversing the sign of $x$ does not affect $P(x)$, and so $P$ is an even function. □
Example 1.5.2. Let \( y(x) = \cos(nx) \) where \( n \in \mathbb{Z} \). Then \( y \) is an even function.

A function \( y \) is said to be an odd function when

\[ y(-x) = -y(x) \text{ for all values of } x \]

Example 1.5.3. Let \( P(x) \) be the polynomial

\[ p_0 + p_1 x + \ldots + p_m x^m \]

If the coefficients \( p_0, p_2, p_4, \ldots \) of even powers of \( x \) in \( P(x) \) are all 0, then reversing the sign of \( x \) does the same to \( P(x) \). So \( P \) is an odd function.

Example 1.5.4. Let \( y(x) = \sin(nx) \) where \( n \in \mathbb{Z} \). Then \( y \) is an odd function.

The simple facts proved in Exercises 1.5.1 1.5.2, 1.5.3, 1.5.4 and 1.5.5, can be put together to simplify the calculation of \( FS \) when \( y \) is an even function.

Proposition 1.5.1. Let \( y : [-\pi, \pi) \to \mathbb{R} \) be an even function. Then

\[ FS(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos(2x) + \ldots a_m \cos(mx) \]

where for \( 0 \leq n \leq m \)

\[ a_n = \frac{2}{\pi} \int_0^\pi y(x) \cos(nx)dx \]

Proof:

Because \( y \) is even then \( y(x)\sin(nx) \) is odd, by Exercises 1.5.4 and 1.5.1. Therefore, by Exercise 1.5.5 and formula (1.10), \( b_n = 0 \) for \( 1 \leq n \leq m \).

Similarly, \( y(x)\cos(nx) \) is even. Then by Exercise 1.5.4 and formula (1.9), \( a_n \) can be worked out as stated above.

Similar simplifications of the calculations for \( FS \) are possible when \( y \) is odd.

Proposition 1.5.2. Let \( y : [-\pi, \pi) \to \mathbb{R} \) be an odd function. Then

\[ FS(x) = b_1 \sin x + b_2 \sin(2x) + \ldots b_m \sin(mx) \]

where for \( 1 \leq n \leq m \)

\[ b_n = \frac{2}{\pi} \int_0^\pi y(x) \sin(nx)dx \]
CHAPTER 1. TRIGONOMETRIC APPROXIMATIONS

Exercises

Exercise 1.5.1. Let $y$ be an even function and $z$ an odd function. Show that $yz$ is an even function. □

Exercise 1.5.2. Let $y$ and $z$ both be even functions. Show that $yz$ is also even. □

Exercise 1.5.3. Let $y$ and $z$ both be odd functions. Show that $yz$ is even. □

Exercise 1.5.4. If $y$ is an even function show that
\[ \int_{-\pi}^{\pi} y(x)dx = 2 \int_{0}^{\pi} y(x)dx \] □

Exercise 1.5.5. If $y$ is an odd function show that
\[ \int_{-\pi}^{\pi} y(x)dx = 0 \] □

Exercise 1.5.6. Prove Proposition 1.5.2. □

Solutions
1.6 Examples

Exponential sums are useful for proving things. Fourier sums are more often used for calculations.

**Example 1.6.1.** Define \( y : [-\pi, \pi) \to \mathbb{R} \) by

\[
y(x) = x^2
\]

Then \( y \) is an even function, so that \( b_n = 0 \) for all \( n \) and

\[
a_n = \frac{2}{\pi} \int_0^\pi y(x) \cos(nx) dx
\]
according to Proposition 1.5.1. Therefore
\[ a_0 = \frac{2}{\pi} \left[ \frac{x^3}{3} \right]_0^\pi = \frac{2\pi^2}{3} \]

For \( 1 \leq n \leq m \) we can integrate by parts to get
\[ a_n = \frac{2}{\pi} \left( [x^2 \sin(nx)]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx) \, dx \right) = \]
\[ \frac{-4}{n\pi} \int_0^\pi x \sin(nx) \, dx \]

A second integration by parts gives
\[ a_n = \frac{-4}{n} \left( [-x \frac{\cos(nx)}{n}]_0^\pi + \frac{1}{n} \int_0^\pi \cos(nx) \, dx \right) = \]
\[ (-1)^n \frac{4\pi}{n^2} \]

Therefore
\[ x^2 \approx \frac{\pi^2}{3} - 4(\cos x - \frac{1}{4} \cos(2x) + \frac{1}{9} \cos(3x) - \frac{1}{16} \cos(4x) + \ldots) \]

Taking \( x = 0 \) we find
\[ \frac{\pi^2}{12} \approx 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \frac{1}{64} + \ldots \]

The left hand side is approximately 0.8224670336. Taking the first 8 terms on the right we get 0.8279621756.

Taking \( x = \frac{\pi}{2} \) we find
\[ \frac{\pi^2}{48} \approx \frac{1}{4} - \frac{1}{16} + \frac{1}{36} - \frac{1}{64} + \ldots \]

namely
\[ \frac{\pi^2}{12} \approx 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots \]

which is the same estimate as before. \( \square \)

**Example 1.6.2.** Define \( y \) by \( y(x) = -1, 0, 1 \) according as \( x \) is less than 0,0 or greater than 0. Although \( y \) is not continuous, it is integrable and we can try to approximate it by a Fourier sum.
Now $y$ is an even function, so that $a_n = 0$ and
\[
 b_n = \frac{2}{\pi} \int \sin(nx) dx = \frac{2}{n\pi}(1 - (-1)^n)
\]
according to Proposition 1.5.1. Therefore for $x \in (0, \pi)$
\[
 \frac{\pi}{4} \approx \sin x + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \frac{1}{7} \sin(7x) + \ldots
\]
It’s remarkable that the expression on the right appears to depend on $x$, whereas the left hand side is constant.
Taking $x = \frac{\pi}{2}$ we obtain
\[
 \frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots
\]
\[\square\]

**Example 1.6.3.** Define $y : [-\pi, \pi) \to \mathbb{R}$ by
\[
 y(x) = x^2 \text{ or } -x^2
\]
according as $x < 0$ or $x \geq 0$. Then $y$ is continuous, and $y$ is an even function so that $a_n = 0$ for all $n$ and
\[
 b_n = \frac{2}{\pi} \int_0^\pi x^2 \sin(nx) dx
\]
Integrating by parts, $b_n$ becomes
\[
 \frac{2}{\pi} (-\frac{x^2 \cos(nx)}{n})_0^\pi + \int_0^\pi \frac{2x \cos(nx)}{n} dx =
\]
\[
 -(-1)^n \frac{2\pi}{n} - \frac{4}{\pi n^2} \int_0^\pi \sin(nx) dx =
\]
\[
 -(-1)^n \frac{2\pi}{n} + \frac{4}{\pi n^3}((-1)^n - 1)
\]
Therefore, for $x \in (0, \pi)$
\[
 x^2 \approx 2\pi(\sin x - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \ldots +)
\]
\[
 -\frac{8}{\pi}(\sin x + \frac{1}{3^3} \sin(3x) + \frac{1}{5^3} \sin(5x) + \ldots )
\]
Taking $x = \frac{\pi}{2}$,

\[
\frac{\pi^2}{4} \approx 2\pi\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right)
- \frac{8}{\pi}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \right)
\]

Using Example 1.6.2 (or Exercise 1.6.2) we obtain

\[
\frac{\pi^2}{4} \approx \frac{\pi^2}{2} - \frac{8}{\pi}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \right)
\]

so that

\[
\frac{\pi^3}{32} \approx 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots
\]

The left hand side is 0.9689461466. Taking the first 5 terms on the right we obtain 0.9694192532. \qed

In order to take this sort of thing further there are two more ingredients.

- abstract linear algebra, and
- convergence of sequences and series.

Each ingredient requires a chapter.
1.6. EXAMPLES

Exercises

Exercise 1.6.1. Take \( x = \frac{\pi}{4} \) and proceed as in Example 1.6.1.

Exercise 1.6.2. For \( |x| < 1 \) expand \( \frac{1}{1+x^2} \) as the geometric series

\[
1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots
\]

and integrate both sides. Take limits as \( x \to 1^- \), and compare your answer with Example 1.6.2.

Exercise 1.6.3. Experiment with Fourier sums of \( y = x^3 \) and see what identities you can prove.

Solutions

Solution 1.6.1.

Solution 1.6.2.

Solution 1.6.3.
Chapter 2

Vector Spaces of Functions

Perhaps you could see, from some of the examples in Chapter 1, that we sometimes need to form infinite sums

- either of numbers, or
- of functions.

For instance, in Example 1.6.1 we met infinite sums of both sorts.

Infinite sums are usually called infinite series and there are technical difficulties in dealing with them. Even very great mathematicians made errors with infinite series at the time that the concepts were being developed. Of course progress is all about learning from mistakes.

Since the machinery for defining infinite sums is fairly elaborate, as in Chapter 3, it saves time and energy to use the same concepts for infinite sums of different kinds of objects, all of which are needed in engineering applications. The way to do this is to have a language in which

- real or complex numbers,
- vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, and
- functions such as $\cos(nx), \sin(nx)$, and $x^n$

are all treated in the same way. These things become objects in abstract vector spaces. An (abstract\footnote{We drop this qualification from now on.}) vector space is just a set with operations which satisfy some simple rules, modelled on rules for addition and scalar multiplication for vectors in $\mathbb{R}^n$. Apart from $\mathbb{R}^n$ and $\mathbb{C}^n$ there are vector spaces of functions. Vector spaces of functions differ from $\mathbb{R}^n$ in several ways:
1. they are usually infinite-dimensional,
2. there is usually much more choice in defining concepts of distances and angles for an infinite-dimensional vector space, and because of this
3. special care is needed with definitions of limits and convergence

Whether an infinite series adds up is a question of convergence, as seen later in Chapter 3 where, because of (3) above, we need to carefully extend the concept of a limit using unfamiliar notions of distance. For example you will have to get used to the idea of sometimes measuring angles and distances between functions using integrals, and that’s what the present Chapter is for.

2.1 Vector Spaces

A real vector space is a set $V$ together with operations

$$+ : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{R} \times V \to V$$

called vector addition and scalar multiplication, satisfying

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. for some $0 \in V$ and all $v \in V$, $v + 0 = v$
4. $0.v = 0$
5. $1.v = v$
6. $r.(u + v) = r.u + r.v$
7. $(r + s).v = r.v + s.v$
8. $(rs).v = r.(s.v)$

Here $u, v, w \in V$ and $r, s \in \mathbb{R}$.

A complex vector space is a set $V$ with operations satisfying the same conditions, except that scalar multiplication is a function

$$\mathbb{C} \times V \to V$$
and the scalars \( r, s \) are permitted to be complex numbers. The field of scalars \( \mathcal{F} \) of a vector space \( V \) is \( \mathbb{R} \) for a real vector space and \( \mathbb{C} \) for a complex vector space. Vector spaces with other fields of scalars are not considered in this Chapter.

**Example 2.1.1.** Let \( V \) be a set containing a single element. Denote this element by \( 0 \) and define

\[
0 + 0 = 0
\]

and

\[
 r \cdot 0 = 0 \text{ for all } r \in \mathcal{F}
\]

Then \( V \) is a vector space with field of scalars \( \mathcal{F} \), called a trivial vector space.

**Exercise 2.1.1.** Prove that if \( V \) is a vector space then, for any \( v \in V \) there is a unique \( w \in V \) with the property that \( v + w = 0 \). (Usually \( w \) is called \( -v \).) \( \square \)

As the name suggests, the sets of ordinary vectors in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) are vector spaces, with respect to the usual operations of vector addition and scalar multiplication.

**Example 2.1.2.** For \( n \geq 1 \) let \( V \) be the space \( \mathbb{R}^n \) of all ordered \( n \)-tuples \((v_1, v_2, \ldots, v_n)\) of real numbers \( v_1, v_2, \ldots, v_n \in \mathbb{R} \). Vector addition of two such \( n \)-tuples \( v, w \) is defined by \( v + w = u \) where

\[
 u_1 = v_1 + w_1, u_2 = v_2 + w_2, \ldots, u_n = v_n + w_n
\]

Scalar multiplication is defined by

\[
 r \cdot v = (rv_1, rv_2, \ldots, rv_n) \in \mathbb{R}^n
\]

The vector space \( \mathbb{R}^1 \) is the set of real numbers, with the usual operations of addition and multiplication. When \( n = 2 \) or \( 3 \) vectors in \( \mathbb{R}^n \) are used to represent translations, velocities and forces in elementary mechanics. Then vector addition and scalar multiplication have natural interpretations. For instance

- \( v + w \) corresponds to the composite of the translations represented by \( v \) and \( w \)

- \( r \cdot v \) corresponds to a translation in the same direction as \( v \) magnified in length by the scalar \( r \) (when \( r < 0 \) the direction is reversed)
Example 2.1.3. For \( n \geq 1 \) let \( V \) be the space \( \mathbb{C}^n \) of all ordered \( n \)-tuples \((v_1, v_2, \ldots, v_n)\) of complex numbers \( v_1, v_2, \ldots, v_n \in \mathbb{C} \). Vector addition of two such \( n \)-tuples \( v, w \) is defined by \( v + w = u \) where
\[
v + w = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)
\]
and
\[
r.v = (rv_1, rv_2, \ldots, rv_n) \in \mathbb{C}^n
\]
The vector space \( \mathbb{C}^1 \) is just the set \( \mathbb{C} \) of complex numbers.

Example 2.1.4. For \( m \geq 1 \) let \( \mathbb{R}P_m[x] \) (respectively \( \mathbb{C}P_m[x] \)) be the set of all polynomials
\[
P(x) = p_0 + p_1x + p_2x + \ldots + p_mx^m
\]
of polynomials of degree \( \leq m \) in a variable \( x \), and with real (respectively complex) coefficients
\[
p_0, p_1, p_2, \ldots, p_m
\]
Vector addition and scalar multiplication are defined as in Example 2.1.4.

Example 2.1.5. Let \( \mathbb{R}P[x] \) (respectively \( \mathbb{C}P[x] \)) be the set of all polynomials
\[
P(x) = p_0 + p_1x + p_2x + \ldots + p_mx^m
\]
in a variable \( x \), and with real (respectively complex) coefficients
\[
p_0, p_1, p_2, \ldots, p_m
\]
Here \( m \) depends on the polynomial.
Vector addition and scalar multiplication are defined as in Example 2.1.4.

Example 2.1.6. Let \( \mathbb{R}F[-\pi, \pi] \) (respectively \( \mathbb{C}F[-\pi, \pi] \)) be the set of real-valued (respectively complex-valued) functions on \([-\pi, \pi]\). (This domain suits the applications we have in mind, but any nonempty subset of \( \mathbb{R} \) would do to give an example of a vector space.)
Vector addition and scalar multiplication are defined by
\[
(y + z)(x) = y(x) + z(x) \text{ and } (r.y)(x) = r(y(x))
\]
Lemma 2.1.1. Let $V$ be a vector space with field of scalars $\mathcal{F}$. Let $W$ be any nonempty subset of $V$ that is closed with respect to vector addition and scalar multiplication, namely

$$v + w \in W \text{ and } r \cdot v \in W$$

whenever $v, w \in W$ and $r \in \mathcal{F}$. Then $W$ is a vector space with respect to the same operations of vector addition and scalar multiplication.

Exercise 2.1.2. Prove Lemma 2.1.1.

A subset $W$ of a vector space $V$ satisfying the conditions of Lemma 2.1.1 is called a vector subspace of $V$.

Example 2.1.7. Let $V$ be a vector space. Then

- the trivial vector subspace $\{0\}$ of $V$ is a vector subspace of $V$, and
- $V$ is a vector subspace of $V$

Exercise 2.1.3. Show that the only vector subspace of $\mathbb{R}$ other than $\mathbb{R}$ itself is the trivial subspace $\{0\}$. Show that, for any $m > 1$, there are infinitely many vector subspaces of $\mathbb{R}^m$.

Example 2.1.8. For $m > 1$ and even, let $W \subset \mathbb{R}P[x]$ (respectively $W \subset \mathbb{C}P[x]$) be the set of all real (respectively complex) polynomials

$$P(x) = p_0 + p_2 x + p_4 x^4 + \ldots + p_m x^m$$

of polynomials of degree $\leq m$ whose odd-numbered coefficients are all $0$. Then $W$ is a vector subspace of $\mathbb{R}P[x]$ (respectively $\mathbb{C}P[x]$).

Example 2.1.9. Let $\mathbb{R}C[-\pi, \pi]$ (respectively $\mathbb{C}C[-\pi, \pi]$) be the set of real-valued (respectively complex-valued) continuous functions on $[-\pi, \pi]$. Then $\mathbb{R}C[-\pi, \pi]$ is a vector subspace of $\mathbb{R}F[-\pi, \pi]$ and $\mathbb{C}C[-\pi, \pi]$ is a vector subspace of $\mathbb{C}F[-\pi, \pi]$.

Example 2.1.10. $\mathcal{F}P_m[x]$ is a vector subspace of $\mathcal{F}P[x]$, which in turn is a vector subspace of $\mathcal{F}C[-\pi, \pi]$.

Example 2.1.11. The set of all integrable real (respectively complex) -valued functions on $[-\pi, \pi]$ is a vector subspace of $\mathbb{R}F[-\pi, \pi]$ (respectively $\mathbb{C}F[-\pi, \pi]$).
Example 2.1.12. The set of all differentiable real (respectively complex) \( -\pi, \pi \) valued functions on \( [-\pi, \pi] \) is a vector subspace of \( \mathbb{R}F[-\pi, \pi] \) (respectively \( \mathbb{C}F[-\pi, \pi] \)).

Example 2.1.13. The even (respectively odd) continuous \( \mathcal{F} \)-valued functions comprise vector subspaces of \( \mathcal{F}C[-\pi, \pi] \).

2.2 Linear Combinations

Let \( V \) be a vector space with field of scalars \( \mathcal{F} \) (either \( \mathbb{R} \) or \( \mathbb{C} \)). Let \( v^1, v^2, \ldots, v^p \) be elements of \( V \) and let \( r_1, r_2, \ldots, r_p \in \mathcal{F} \) be scalars. Then

\[
    r_1 v^1 + r_2 v^2 + \ldots + r_p v^p \in \mathcal{F}
\]

is said to be a linear combination of \( v^1, v^2, \ldots, v^p \) with coefficients \( r_1, r_2, \ldots, r_p \).

Let \( S \) be a nonempty subset of \( V \). The set \( <S> \) of all linear combinations of elements of \( V \) (with \( p \) arbitrary, and arbitrary coefficients in \( \mathcal{F} \)) is called the span of \( S \). Define the span \( <\emptyset> \) of the empty set \( \emptyset \subset V \) to be the trivial vector subspace \{0\} of \( V \). If you have trouble proving the following result, ask your lecturer or tutor.

Proposition 2.2.1. \( <S> \) is a vector subspace of \( V \).

When \( <S> \supseteq V \) the vector space \( V \) is said to be spanned by \( S \).

Example 2.2.1. Let \( S \subset \mathbb{R}F[-\pi, \pi] \) be

\[
    \{\cos(nx), \sin(x) : n \in \mathbb{Z} \text{ with } n \geq 0\}
\]

Then \( <S> \) is the vector space of real-valued functions which can be written in the form

\[
    \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos(nx) + b_n \sin(nx)
\]

for some \( m \in \mathbb{Z}_+ \).

Example 2.2.2. Let \( S \subset \mathbb{C}F[-\pi, \pi] \) be

\[
    \{e^{inx} : n \in \mathbb{Z}\}
\]

Then \( <S> \) is the vector space of complex-valued functions of the form

\[
    \sum_{n=-m}^{m} c_n e^{inx}
\]

for some \( m \in \mathbb{Z}_+ \).
2.2. LINEAR COMBINATIONS

Definition 2.2.1. $S$ is said to be linearly independent when, given distinct $v^1, v^2, \ldots v^p \in S$

the only scalars $r_1, r_2, \ldots r_p \in \mathcal{F}$ for which

$$r_1 v^1 + r_2 v^2 + \ldots r_p v^p = 0$$

are

$$r_1 = r_2 = \ldots = r_p = 0$$

Exercise 2.2.1. Check that $\emptyset \subset V$ is linearly independent.

Exercise 2.2.2. Let $v \in V$. Prove that $\{v\}$ is linearly independent if and only if $v = 0$.

Exercise 2.2.3. Prove that any subset of a linearly independent set is also linearly independent.

Exercise 2.2.4. Let $S$ be any subset of $V$ and let $w \in S >$ where $w \notin S$. Prove that $S$ is not linearly independent.

Exercise 2.2.5. Let $S$ be any subset of $V$. Prove that $S \cup \{0\}$ is not linearly independent.

Exercise 2.2.6. Fix $m \in \mathbb{Z}_+$, and for $1 \leq n \leq m$ let $e^n$ be the vector whose coordinate in position $j$ is $\delta^n_j$. Prove that

$$\{e^1, e^2, \ldots e^m\}$$

is linearly independent.

Exercise 2.2.7. Prove that the set $S$ in Example 2.2.1 is linearly independent. (Hint: if a linear combination is $0$ try working out its Fourier coefficients $a_n, b_n$.)

Exercise 2.2.8. Prove that the set $S$ in Example 2.2.2 is linearly independent.

Exercise 2.2.9. Prove that the set

$$\{e^{mx} : m \in \mathbb{Z}\}$$

is linearly independent.

(Hint: start with the coefficient $r_n$ of $e^{nx}$ where $n$ is largest. If $r_n \neq 0$ then for sufficiently large values of $x$ the term $r_n e^{nx}$ dominates the others.)
CHAPTER 2. VECTOR SPACES OF FUNCTIONS

**Definition 2.2.2.** A linearly independent subset $S$ of a vector space $V$ is called a *basis* of $V$ when $S$ spans $V$. □

**Exercise 2.2.10.** Prove that $\emptyset$ is a basis of the trivial vector space $\{0\}$. □

**Exercise 2.2.11.** Prove that the set

$$S = \{e^1 = (1, 0, 0, \ldots 0), e^2 = (0, 1, 0, \ldots 0), \ldots e^m = (0, 0, 0, \ldots 1)\}$$

in Example 2.2.2 is a basis of $\mathbb{R}^m$. $S$ is called the *standard basis* of $\mathbb{R}^m$. □

**Exercise 2.2.12.** Prove that

$$\{x^n : 0 \leq n \leq m\}$$

is a basis of $\mathbb{R}P_m[x]$. □

A nontrivial vector space over $\mathbb{R}$ or $\mathbb{C}$ either has no basis at all, or many bases.

**Exercise 2.2.13.** Show that every basis of $\mathbb{R}^1$ has the form

$$\{v\}$$

where $v \in \mathbb{R}$ is nonzero. Show that every such set is a basis of $\mathbb{R}^1$. □

**Exercise 2.2.14.** Show that every basis of $\mathbb{R}^2$ has the form

$$\{(a, b), (c, d)\}$$

where $ad - bc \neq 0$. Show that every such set is a basis of $\mathbb{R}^2$.

(*Hint: consider the system*

$$r_1a + r_2c = d_1 \quad (2.1)$$

$$r_1b + r_2d = d_2 \quad (2.2)$$

*of two linear equations in two unknowns $r_1, r_2 \in \mathbb{R}$, where $a, b, c, d, d_1, d_2 \in \mathbb{R}$ are given.*) □

**Exercise 2.2.15.** Show that every basis of $\mathbb{R}^3$ has the form

$$\{(a, b, c), (d, e, f), (g, h, i)\}$$

where $a(ei - fh) - b(ai - fg) + c(dh - eg) \neq 0$. Show that every such set is a basis of $\mathbb{R}^3$. 
(Hint: Let \( a, b, c, d, e, f, g, h, i, d_1, d_2, d_3 \in \mathbb{R} \) be given, and consider the system
\[
\begin{align*}
    r_1a + r_2d + r_3g &= d_1 \\
    r_1b + r_2e + r_3h &= d_2 \\
    r_1c + r_2f + r_3i &= d_3
\end{align*}
\]
(2.3)
(2.4)
(2.5)
of three linear equations in three unknowns \( r_1, r_2, r_3 \in \mathbb{R} \).)

It’s a similar story for complex vector spaces.

**Example 2.2.3.** The set
\[ S = \{ e^1 = (1, 0, 0, \ldots 0), e^2 = (0, 1, 0, \ldots 0), \ldots e^m = (0, 0, 0, \ldots 1) \} \]
is a basis of \( \mathbb{C}^m \). \( S \) is called the standard basis of \( \mathbb{C}^m \).

A basis of a vector space need not be finite.

**Exercise 2.2.16.** Let \( V \) be the vector subspace of \( \mathbb{R}F[-\pi, \pi] \) consisting of functions \( y \) where \( y(x) = 0 \) for all but finitely many values of \( x \). For \( x \in [-\pi, \pi] \) define
\[ \delta_x : [-\pi, \pi] \to \mathbb{R} \]
by
\[ \delta_x(t) = 1 \text{ for } t = x, \text{ and } \delta_x(t) = 0 \text{ for } t \neq x \]
Prove that
\[ \{ \delta_x : x \in [-\pi, \pi] \} \]
is a basis of \( V \).

**Proposition 2.2.2.** Suppose that \( V \) does have a finite basis, with \( m \) elements. Then

1. every basis of \( V \) is finite with \( m \) elements, and

2. every linearly independent subset of \( V \) with \( m \) elements is a basis of \( V \).

**Definition 2.2.3.** A vector space \( V \) is said to be finite-dimensional when it has a finite basis; otherwise \( V \) is infinite-dimensional. The dimension \( \text{dim}(V) \) of a vector space \( V \) is

- \( \infty \) when \( V \) does not have a finite basis, and
• $m$ when $V$ has a basis with $m$ elements.

A corollary is a result that’s an easy consequence of a proposition or theorem, in this case Proposition 2.2.2.

**Corollary 2.1.** Let $W$ be a vector subspace of a finite-dimensional vector space $V$. Then $W = V$ if and only if $\dim(W) = \dim(V)$. □

**Exercise 2.2.17.** Let $V$ be a vector space. Prove that $\dim(V) = 0$ if and only if $V = \{0\}$. □

**Exercise 2.2.18.** Prove that $\dim(\mathbb{R}^m) = m = \dim(\mathbb{C}^m)$. □

**Exercise 2.2.19.** Prove that $\dim(P_m[x]) = m + 1$. □

**Exercise 2.2.20.** Prove that $\dim(<S>) = \infty$ where $S$ is the set in either Example 2.2.1 or Example 2.2.2. □

### 2.3 Linear Transformations

Let $V$ and $W$ be vector spaces with the same field of scalars $\mathcal{F}$. A linear transformation from $V$ to $W$ is a function

$$L : V \to W$$

which preserves linear combinations. More precisely, the requirement is that

$$L(r_1v^1 + r_2v^2 + \ldots + r_pv^p) = r_1L(v^1) + r_2L(v^2) + \ldots + r_pL(v^p)$$

for any $v^1, v^2, \ldots, v^p \in V$ and any $r_1, r_2, \ldots, r_p \in \mathcal{F}$. Here $p \in \mathbb{Z}_+$. □

**Exercise 2.3.1.** Prove that it is enough to state the requirement for $p = 2$. □

Let $L : V \to W$ be a linear transformation.

• The **range** of $L$ is defined to be

$$\text{Range}(L) = \{L(v) : v \in V\} \subset W$$

• The **kernel** of $L$ is

$$\text{Ker}(L) = \{v \in V : L(v) = 0\} \subset V$$
Exercise 2.3.2. Let \( L : V \to W \) be a linear transformation. Prove that \( \text{Range}(L) \) and \( \text{Ker}(L) \) are vector subspaces of \( W \) and \( V \) respectively. 

A projection is a linear transformation \( L \) with the property that
\[
L = L \circ L : V \to V
\]
Then the projection is said to be onto the vector subspace \( \text{Range}(L) = L(V) \) of \( V \).

Example 2.3.1. Fix \( 1 \leq n < m \) and define \( L : \mathbb{R}^m \to \mathbb{R}^m \) by
\[
L(v_1, v_2, \ldots, v_m) = (v_1, v_2, \ldots, v_n, 0, 0, \ldots 0)
\]
Then \( L \) is a projection from \( \mathbb{R}^m \) onto \( \mathbb{R}^n \times \{0\} \cong \mathbb{R}^n \). 

Exercise 2.3.3. Fix \( 1 \leq n < m \). Show that there are many projections from \( \mathbb{R}^m \) onto \( \mathbb{R}^n \times \{0\} \). (Hint: consider orthogonal projections onto other vector subspaces of \( \mathbb{R}^m \)).

Exercise 2.3.4. Let \( L : V \to W \) be a linear transformation. Prove that \( L(0) = 0 \).
(Hint: consider \( L(0) \).) 

Exercise 2.3.5. Let \( L, M : V \to W \) be linear transformations, where \( V, W \) are vector spaces over the same field of scalars \( F \). Let \( r \in F \). Prove that

1. \( L + M : V \to W \) given by
\[
(L + M)(v) = L(v) + M(v)
\]
is a linear transformation, and

2. \( rL : V \to W \) given by
\[
(rL)(v) = r(L(v))
\]
is a linear transformation,

3. the set \( L(V, W) \) of all linear transformations from \( V \) to \( W \) is also a vector space over \( F \), with respect to the operations + and . defined in (1), (2).
Exercise 2.3.6. prove that a composite of linear transformations is a linear transformation.

Exercise 2.3.7. Prove that any linear transformation $L : \mathbb{R} \to \mathbb{R}$ has the form

$$L(x) = ax$$

where $x \in \mathbb{R}$

for some fixed $a \in \mathbb{R}$.

(Hint: set $a = L(1)$.)

Exercise 2.3.8. Prove that any linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ has the form

$$L(x, y) = ax + by$$

where $x, y \in \mathbb{R}$ for some fixed $a, b \in \mathbb{R}$. What can you say about linear transformations

- from $\mathbb{R}^n$ to $\mathbb{R}$?
- from $\mathbb{R}^n$ to $\mathbb{R}^m$?

The next example is important for future work on approximations by trigonometric sums.

Example 2.3.2. Let $V = W = FC[−\pi, \pi]$ where $F$ is $\mathbb{R}$ or $\mathbb{C}$. Fix $m > 0$ and let

$$F : V \to W$$

be the transformation mapping $y \in FC[−\pi, \pi]$ to its approximation

$$FS \in W \text{ or } ES \in W$$

according as $F$ is $\mathbb{R}$ or $\mathbb{C}$ respectively.

Proposition 2.3.1. The transformations $L$ defined in Example 2.3.2 are linear transformations.

Proof:

(only for the case $F = \mathbb{C}$) Let $y, z \in \mathbb{C}C[−\pi, \pi]$. Then

$$F(y)(x) = \sum_{n=-m}^{m} c_n e^{nix} \text{ and } F(z)(x) = \sum_{n=-m}^{m} d_n e^{nix}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-nix} dx \text{ and } d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} z(x) e^{-nix} dx$$
according to the formula in Proposition 1.8. Similarly, for \(a, b \in \mathbb{C}\)
\[
F(ay + bz)(x) = \sum_{n=-m}^{m} e_n e^{nix}
\]
where
\[
e_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (ay(x) + bz(x))e^{-nix}dx = ac_n + bd_n
\]
It follows that
\[
F(ay + bz) = aF(y) + bF(z)
\]
and the proposition is proved when \(\mathcal{F} = \mathbb{C}\). The case \(\mathcal{F} = \mathbb{R}\) is similar.

## 2.4 Linear Isomorphisms

Let \(L : V \rightarrow W\) be a linear transformation. If \(L\) is a linear transformation then

- the dimension of \(\text{Range}(L)\) is called the rank \(\rho(L)\) of \(L\), and
- the dimension of \(\text{Ker}(L)\) is the nullity \(\nu(L)\) of \(L\).

Recall that a function is

- **one-to-one** when it maps distinct elements of its domain to distinct elements of its range,
- **onto** when its range is the whole of its target space,
- **bijective** when it is one-to-one and onto.

**Example 2.4.1.** Let \(W\) be a vector subspace of a vector space \(V\). Then the inclusion

\[
W \rightarrow V \quad \text{of} \ W \quad \text{in} \ V
\]
given by \(x \mapsto x\) for \(x \in W\)

is one-to-one.

**Proposition 2.4.1.** Let \(L : V \rightarrow W\) be a linear transformation. Then

1. \(L\) is one-to-one if and only if \(\nu(L) = 0\), and
2. if \(\dim(W) < \infty\) then \(L\) is onto if and only if \(\rho(L) = \dim(W)\)
Proof:
If \( L \) is one-to-one and \( L(v) = 0 \) then \( v = 0 \in V \) because \( L(0) = 0 \) also. Therefore \( \text{Ker}(L) = \{0\} \).

Conversely, suppose \( \text{Ker}(L) = \{0\} \) and that \( L(v) = L(w) \) where \( v, w \in V \).
Because \( L \) is linear, \( L(v - w) = L(v) - L(w) = 0 \), namely \( v - w \in \text{Ker}(L) \).
Therefore \( v - w = 0 \) and \( v = w \).
So \( L \) is one-to-one if and only if \( \text{Ker}(L) = \{0\} \). The proposition now follows from Exercise 2.2.17 and Corollary 2.1.

Part (2) of Proposition 2.4.1 is not always true when \( W \) is infinite-dimensional.

Example 2.4.2. Let \( W = \mathbb{R}F[-\pi, \pi] \) and let \( L \) be the inclusion of the subspace \( V = \mathbb{R}C[-\pi, \pi] \) in \( W \). Then \( \rho(L) = \infty \) but \( L(V) = V \neq W \).

Another basic result is the Rank-Nullity Theorem:

Theorem 2.2. Let \( L : V \rightarrow W \) be a linear transformation. Then

\[
\rho(L) + \nu(L) = \dim(V)
\]

A linear transformation \( L : V \rightarrow W \) is a linear isomorphism when it has a linear inverse, namely when there is a linear transformation \( M : W \rightarrow V \) such that

\[
M \circ L : V \rightarrow V \quad \text{and} \quad L \circ M : W \rightarrow W
\]

are the identity transformations \( 1_V \) and \( 1_W \) on \( V \) and \( W \) respectively. When there is a linear isomorphism from \( V \) to \( W \), \( V \) and \( W \) are said to be isomorphic, \( V \cong W \). From Theorem 2.2 we have

Corollary 2.3. If \( V \cong W \) then \( \dim(V) = \dim(W) \).

Exercise 2.4.1. Prove that a composite of linear isomorphisms is a linear isomorphism.

Exercise 2.4.2. Prove that \( \cong \) is an equivalence relation on the set of all vector spaces \( U, V, W, \ldots \) over the same field of scalars \( \mathcal{F} \), namely that

- \( V \cong V \),
- if \( V \cong W \) then \( W \cong V \), and
- if \( U \cong V \) and \( V \cong W \) then \( U \cong W \).
Exercise 2.4.3. Let $V$ and $W$ be vector spaces of the same finite dimension $m$ over the same field of scalars $F$. Prove that $V \cong W$.
(Hint: show that $V \cong F^m$.)

Exercise 2.4.4. Prove that $P_m[x] \cong \mathbb{R}^{m+1}$.

2.5 Matrices

In practice calculations with linear transformations $L : V \to W$ are usually carried out using matrices, at least when $V$ and $W$ are finite-dimensional. Here’s how.

First we choose bases $\{e^1, e^2, \ldots, e^n\}$ and $\{f^1, f^2, \ldots, f^m\}$ of $V$ and $W$ respectively. When $V$ and $W$ are $\mathbb{R}^n$ and $\mathbb{R}^m$ it’s common to choose the standard bases, but this is not compulsory. Then for each $1 \leq j \leq n$ the vector $L(e^j)$ is expressible as a linear combination

$$A_{1j} f^1 + A_{2j} f^2 + \ldots + A_{mj} f^m$$

where the $a_{ij}$ are elements of the field of scalars $F$. Form the rectangular array

$$[L] = A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\
A_{21} & A_{22} & A_{23} & \ldots & A_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & A_{m3} & \ldots & A_{mn}
\end{bmatrix}$$

of elements of $F$ where $a_{ij}$ appears in the $i$th row and $j$th column. Then $[L]$ has $m$ rows and $n$ columns, and is called an $m \times n$ matrix over $F$. Provided the bases $\{e^1, e^2, \ldots, e^n\}$ and $\{f^1, f^2, \ldots, f^m\}$ are given, all information about the linear transformation $L$ is recorded in its matrix $[L] = A$.

An $m \times n$ matrix is just a rectangular array of scalars with $m$ rows and $n$ columns. The sum of two such matrices $A$ and $B$ is the matrix $A + B$ where

$$(A + B)_{ij} = A_{ij} + B_{ij} \text{ for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

If $r \in F$ then $rA$ is defined to be the $m \times n$ matrix given by

$$(rA)_{ij} = rA_{ij} \text{ for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

Definition 2.5.1. The $m \times n$ zero matrix is the $m \times n$ matrix $0_{m \times n}$ with 0 in every position.
Proposition 2.5.1. Let $A, B, C$ be $m \times n$ matrices over $\mathcal{F}$, and let $r, s \in \mathcal{F}$. Then

1. $A + 0_{m \times n} = A$,
2. $(A + B) + C = A + (B + C)$,
3. $1A = A$,
4. $0A = 0_{m \times n}$
5. $r(A + B) = rA + rB$, 
6. $(r + s)A = rA + sA$

Exercise 2.5.1. Prove Proposition 2.5.1

The matrix product is a much less intuitive operation. For a start, it matters which order you multiply matrices in, and the matrix product $AB$ is defined only when $A$ has the same number of columns as $B$ has rows. So $A$ should be an $m \times n$ matrix and $B$ should be $n \times p$. Then the matrix product $AB$ is defined to be the $m \times p$ matrix given by

$$(AB)_{ik} = A_{i1}B_{1k} + A_{i2}B_{2k} + \ldots A_{in}B_{nk}$$

The reason matrix multiplication is defined in this way is to make the following result hold true.

Proposition 2.5.2. Let $L : V \rightarrow W$ and $M : W \rightarrow U$ be linear transformations, where $U, V, W$ are vector spaces over $\mathcal{F}$ of dimensions $p, n, m$ respectively. Choose bases for $U, V, W$. Then with respect to these bases the matrices of $L, M$ and their composite

$$(M \circ L) : V \rightarrow U$$

satisfy

$$[M \circ L] = [M][L]$$

where multiplication on the right is matrix multiplication.

Exercise 2.5.2. Prove Proposition 2.5.2
Example 2.5.1. Let \( L : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation, and choose the standard bases on \( \mathbb{R}^n \) and \( \mathbb{R}^m \). Then a vector
\[
v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n
\]
can be represented as a column vector
\[
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
\]
with \( n \) rows, namely an \( n \times 1 \) matrix. Likewise, vectors in \( \mathbb{R}^m \) correspond to \( m \times 1 \) matrices.

Then the column vector corresponding to \( L(v) \) is the matrix product
\[
[L]
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
\]

\[\square\]

Definition 2.5.2. The \( n \times n \) identity matrix is the \( n \times n \) matrix \( 1_{n \times n} \) whose entry in row \( i \) and column \( j \) is \( \delta_{ij} \).

\[\square\]

Proposition 2.5.3. Let \( A, B \) be \( m \times n \) matrices over \( F \), let \( C \) be an \( n \times p \) matrix over \( F \), let \( D \) be a \( p \times q \) matrix over \( F \), let \( E \) be a \( q \times m \) matrix over \( F \), and let \( r, s \in F \). Then

1. \( A 1_{n \times n} = A = 1_{m \times m} A \),
2. \( (AC)D = A(CD) \),
3. \( (A + B)C = AC + BC \),
4. \( E(A + B) = EA + EB \),
5. \( 0_{q \times m} A = 0_{q \times n} \),
6. \( A 0_{n \times p} = 0_{m \times p} \),
7. \( r(AC) = (rA)C = A(rC) \).
CHAPTER 2. VECTOR SPACES OF FUNCTIONS

If $A$ and $B$ are both $n \times n$ matrices (square matrices of side $n$), then it is very unusual for

$$AB = BA$$

Exercise 2.5.3. Choose two $2 \times 2$ matrices $A$ and $B$ at random. Calculate the matrix products $AB$ and $BA$. Do this until

$$AB \neq BA$$

2.6 Norms

Let $V$ be a vector space with field of scalars $\mathcal{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). A norm on $V$ is a function

$$\| \| : V \rightarrow \mathbb{R}$$

with the properties

1. $\|v\| \geq 0$ for all $v \in V$,
2. $\|v\| = 0$ if and only if $v = 0$,
3. $\|r \cdot v\| = |r| \|v\|$ where $v \in V$ and $r \in \mathcal{F}$,
4. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

Infinitely many different norms can be constructed for any nontrivial vector space.

Exercise 2.6.1. For $V = \mathbb{R}$ or $\mathbb{C}$ prove that any norm on $V$ is given by

$$\|v\| = c|v|$$

for some nonzero constant $c$. □

Of course there are more possibilities in higher dimensional vector spaces.

Exercise 2.6.2. For $V = \mathbb{R}^m$ set

$$\|(v_1, v_2, \ldots, v_m)\| = |v_1| + |v_2| + \ldots + |v_m|$$

and verify that this defines a norm. □
Exercise 2.6.3. For $V = \mathbb{R}^m$ set
\[
\|(v_1, v_2, \ldots v_m)\| = \text{Max}\{|v_n| : 1 \leq n \leq m\}
\]
and verify that this defines a norm.

In practice, however, there is usually a preferred norm. A vector space equipped with a norm is called a **normed vector space**.

**Example 2.6.1.** The **Euclidean norm** on $\mathbb{R}^m$ is given by
\[
\|(v_1, v_2, \ldots v_m)\|^2 = v_1^2 + v_2^2 + \ldots v_m^2
\]
We will put off until Section 2.8 the verification of condition (4)
\[
\|v + w\| \leq \|v\| + \|w\|
\]
known as the **triangle inequality**.

Note, however, that $\|v\|$ is the familiar **Euclidean length** of the vector $v$. The **Euclidean distance** between $v, w \in \mathbb{R}^m$ is then
\[
\|v - w\|
\]

**Example 2.6.2.** The **Hermitian norm** on $\mathbb{C}^m$ is given by
\[
\|(v_1, v_2, \ldots v_m)\|^2 = |v_1|^2 + |v_2|^2 + \ldots |v_m|^2
\]
Again it is better to put off verifying the triangle inequality.

Although the Euclidean norm is the most common one on $\mathbb{R}^m$, it is sometimes convenient to use other norms, such as in Exercises 2.6.2, 2.6.3. Sometimes norms are needed on vector spaces of **functions**. On such spaces there is an even greater range of possible norms. In practice the choice depends on the particular application we have in mind.

**Exercise 2.6.4.** The **uniform norm** on $\mathcal{F}C[-\pi, \pi]$ is defined by
\[
\|y\| = \text{Sup}\{|y(x)| : x \in [-\pi, \pi]\}
\]
Prove that the uniform norm satisfies the conditions for a norm.

An alternative that we shall use more often is...
Example 2.6.3. Let \( V = \mathcal{F}C[-\pi, \pi] \), and for \( y \in V \) define
\[
\|y\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |y(x)|^2 dx
\]
The triangle inequality will be verified in Section 2.8, but the other conditions for a norm are easy to check.

Exercise 2.6.5. Let \( V = \mathcal{F}C[-\pi, \pi] \), and for \( y \in V \) define
\[
\|y\| = \int_{-\pi}^{\pi} |y(x)| dx
\]
Verify that this defines a norm on \( V \).

2.7 Inner Products

Let \( V \) be a vector space with field of scalars \( \mathcal{F} \) (either \( \mathbb{R} \) or \( \mathbb{C} \)). An inner product on \( V \) is a function

\[
\langle \ , \ \rangle : V \times V \to \mathcal{F}
\]
with the properties

1. \( \langle av + bw, u \rangle = a \langle v, u \rangle + b \langle w, u \rangle \) for all \( a, b \in \mathcal{F} \) and all \( v, w \in V \),

2. \( \langle w, v \rangle \) is the complex conjugate of \( \langle v, w \rangle \), and

3. \( \langle v, v \rangle \geq 0 \) for all \( v \in V \),

4. \( \langle v, v \rangle = 0 \) if and only if \( v = 0 \).

Of course when \( \mathcal{F} = \mathbb{R} \) condition (2) above means \( \langle w, v \rangle = \langle v, w \rangle \).

A vector space equipped with an inner product is called an inner product space.

Example 2.7.1. Let \( V = \mathbb{R}^m \). Then the inner product \( \langle \ , \ \rangle \) on \( \mathbb{R}^m \) given by
\[
\langle v, w \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_m w_m
\]
is called the Euclidean inner product.
Example 2.7.2. Let $V = \mathbb{C}^m$. Then the inner product $<\ ,\ >$ on $\mathbb{C}^m$ given by

$$<v, w> = v_1\bar{w}_1 + v_2\bar{w}_2 + \ldots + v_m\bar{w}_m$$

is called the Hermitian inner product. 

Here is something a little more exotic, and very useful for applications with trigonometric sums.

Example 2.7.3. Define an inner product on $\mathcal{F}C[-\pi, \pi]$ by

$$<y, z> = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)\bar{z}(x)dx$$

2.8 Inner Products and Norms

An inner product $<\ ,\ >$ gives rise to an associated norm according to the formula

$$||v||^2 = <v, v>$$

Most of the conditions for a norm follow easily from the definitions. The exception is the triangle inequality (4)

$$||v + w|| \leq ||v|| + ||w||$$

To prove this we need the Cauchy-Schwarz Inequality:

Proposition 2.8.1. Let $v, w \in V$ where $V$ is an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Then

$$|<v, w>| \leq ||v|| ||w||$$

Proof:

If $<v, w>$ happens to be real then we argue as follows. For any $t \in \mathbb{R}$ we have

$$<v - tw, v - tw> \geq 0$$

Expanding the left hand side we obtain a quadratic in the variable $t$, namely

$$<w, w>t^2 - 2<v, w>t + <v, v> \geq 0$$

The discriminant of the quadratic is therefore non-positive:

$$4<v, w>^2 - 4<w, w><v, v> \leq 0$$
CHAPTER 2. VECTOR SPACES OF FUNCTIONS

Therefore
\[<v, w>^2 \leq <v, v><w, w> = \|v\|^2\|w\|^2\]

This proves the Cauchy-Schwarz Inequality, but only in the case where \(<v, w> \in \mathbb{R}\). Write \(\xi = <v, w>\).

When \(\xi \notin \mathbb{R}\) let

\[\tilde{w} = \frac{\xi}{|\xi|}w\]

Then

- \(<v, \tilde{w}> = |<v, w>| \in \mathbb{R}\),
- \(\|\tilde{w}\| = \|w\|\) and
- \(|<v, \tilde{w}>| = |<v, w>|\)

Replacing \(w\) by \(\tilde{w}\) we can apply the argument in the previous case. This proves the Cauchy-Schwarz Inequality in general. \(\square\)

Corollary 2.4. The triangle inequality holds for a norm \(\|\|\) associated with an inner product.

Proof: \(\|v + w\|^2\) is

\[<v + w, v + w> = <v, v> + <v, w> + <w, v> + <w, w> \leq\]

\[<v, v> + 2\|v\|\|w\| + <w, w> = \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2\]

Now take positive square-roots of both sides. \(\square\)

Exercise 2.8.1. Verify that the norms in Examples 2.6.1, 2.6.2, 2.6.3 satisfy the triangle inequality. \(\square\)

For a real vector space, it follows from Proposition 2.8.1 that, for nonzero vectors \(v, w\) in a normed vector space, the quantity

\[\frac{<v, w>}{\|v\|\|w\|}\]

lies in the interval \([-1, 1]\).
Example 2.8.1. When $V = \mathbb{R}^2$ with the Euclidean inner product, $\|v\|$ is the length of the vector $v$, and when $v, w$ are nonzero

$$\frac{\langle v, w \rangle}{\|v\|\|w\|}$$

is the cosine of the angle between $v$ and $w$. □

Not every norm is associated with an inner product.

Exercise 2.8.2. Show that the norms in Exercises 2.6.2, 2.6.3 do not come from an inner product, unless $m = 1$.

(Hint: verify that

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$$

for a norm arising from an inner product.) □

Exercise 2.8.3. Prove that the uniform norm in Exercise 2.6.4 is not associated with an inner product on $\mathcal{FC}[-\pi, \pi]$.

(Hint: compare with the second part of Exercise 2.8.2.) □

2.9 Orthonormal Sets

A subset $S$ of an inner product space is said to be orthonormal when

$$\langle v, w \rangle = \delta^w_v$$

for all $v, w \in S$

Here $\delta^w_v$ is 1 when $v = w$ and 0 otherwise.

Example 2.9.1. Any subset of the standard basis of either $\mathbb{R}^m$ or $\mathbb{C}^m$ is orthonormal. □

Exercise 2.9.1. Prove that an orthonormal subset $S$ of an inner product space $V$ is linearly independent.

(Hint: if $r_1v^1 + r_2v^2 + \ldots + r_mv^m = \mathbf{0}$ try taking inner products with $v_j$ for some $1 \leq j \leq m$.) □

There is a process called Gram-Schmidt Orthogonalization which converts an subset

$$S = \{v^1, v^2, \ldots v^m, \ldots\}$$

of an inner product space $V$ into an orthonormal set with the same span. It goes as follows.
CHAPTER 2. VECTOR SPACES OF FUNCTIONS

- First, any \(0\) vectors are deleted from \(S\). If this leaves nothing stop.

- Otherwise \(v_1\) is replaced by the unit vector \(\frac{1}{\|v_1\}} v_1\) corresponding to \(v_1\).

- For \(n > 1\) delete from \(S\) any vectors \(v_n\) that are multiples of \(v_1\). If this leaves only \(v_1\) stop.

- Otherwise \(v_2\) is not now a multiple of \(v_1\) it follows that
\[
v_2 - <v_2, v_1 > v_1 \neq 0
\]
Replace \(v_2\) by the unit vector corresponding to the left hand side.

- For \(n > 2\) delete from \(S\) any vectors \(v_n\) that are linear combinations of \(v_1, v_2\). If this leaves only \(v_1, v_2\) stop.

- Otherwise replace \(v_3\) by the unit vector corresponding to
\[
v_3 - <v_1, v_3 > v_1 - <v_2, v_3 > v_2 \neq 0
\]

- For \(n > 3\) delete from \(S\) any vectors \(v_n\) that are linear combinations of \(v_1, v_2, v_3\). If this leaves only \(v_1, v_2, v_3\) stop.

and so on. Gramm-Schmidt orthogonalisation is a useful technique, but we won’t meet it again in these notes.

Lemma 1.3.1 can now be restated in terms of the inner product in Example 2.7.3, as follows.

**Proposition 2.9.1.** Let \(< , >\) be the inner product on \(\mathbb{C}[−\pi, \pi]\) defined in Example 2.7.3. Then with respect to this inner product the functions which map \(x\) to
\[
\ldots e^{-m_\infty} x, e^{-(m-1)i}x, \ldots e^{-ix}, 1, e^{ix}, \ldots e^{(m-1)ix}, e^{m_\infty} x, \ldots
\]
constitute an orthonormal set.

Using Lemma 2.9.1, Proposition 1.8 can be reproved more elegantly, as follows.

**Proposition 2.9.2.** Let \(S = \{e_1, e_2, \ldots, e_m\}\) be an orthonormal set in a vector space \(V\) and let \(W\) be the span \(<S>\) of \(S\). If \(y \in W\) then
\[
y = \sum_{n=1}^{m} c_n e_n
\]
where
\[ c_n = \langle y, e^n \rangle \]

Proof:
Because \( y \in S \) we can write it in the form
\[
y = r_1 e^1 + r_2 e^2 + \ldots + r_m e^m
\]
Then for \( 1 \leq n \leq m \)
\[
\langle y, e^n \rangle = \sum_{p=1}^{m} c_p < e^p, e^n > = \sum_{p=1}^{m} c_p \delta_p^n = c_n
\]

\( \square \)

## 2.10 Parseval’s Identity

Let \( V \) be an inner product space. First we prove a generalisation of Pythagoras’ Theorem.

**Proposition 2.10.1.** Let \( \{e^1, e^2, \ldots, e^m\} \) be an orthonormal set and suppose that
\[
v = r_1 e^1 + r_2 e^2 + \ldots + r_m e^m
\]
Then
\[
\|v\|^2 = |r_1|^2 + |r_2|^2 + \ldots + |r_m|^2
\]

Proof:
\[
\|v\|^2 = \langle v, v \rangle = \sum_{j=1}^{m} r_j e^j, \sum_{k=1}^{m} r_k e^k \rangle = \sum_{j=1}^{m} \sum_{k=1}^{m} r_j \bar{r}_k \delta_j^k = \sum_{j=1}^{m} |r_j|^2
\]

\( \square \)

**Example 2.10.1.** Taking \( V = \mathbb{R}^2 \) with the Euclidean inner product, the standard basis \( \{e^1, e^2\} \) is orthonormal. Proposition 2.10.1 then amounts to Pythagoras’ Theorem, which is usually proved geometrically. Our proof is algebraic.

Here is another example. The statement of the next proposition (or rather its generalisation in Chapter 3 to infinite series) is called Parseval’s Identity.
Proposition 2.10.2. For any $m \geq 1$ let $W$ be the vector subspace $< S >$
where
\[ S = \{ e^{-mx}, e^{-(m-1)ix}, \ldots e^{-ix}, 1, e^{ix}, \ldots e^{(m-1)ix}, e^{mx} \} \]
Then for $y \in W$ we have
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |y(x)|^2 dx = |c_{-m}|^2 + |c_{-m+1}|^2 + \ldots + |c_{1}|^2 + \ldots |c_{m-1}|^2 + |c_{m}|^2 \]
where, for $-m \leq j \leq m$ the $c_j$ are given by
\[ c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-jix} dx \]

Proof:
Let $V = \mathbb{C}[\pi, pi]$ with the inner product in Example 2.7.3. Then, by Proposition 2.9.1 and Exercise 2.9.1, $S$ is an orthonormal basis of $W$. Therefore, for any $y \in W$
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |y(x)|^2 dx = < y, y > = \sum_{j=-m}^{m} |c_j|^2 \]

Parseval’s Identity can be used to prove further interesting facts about $\pi$. To do this recall
\[ c_0 = \frac{a_0}{2} \]
\[ c_n = \frac{a_n - ib_n}{2} \text{ and } c_{-n} = \frac{a_n + ib_n}{2} \]
for $1 \leq n \leq m$.

Example 2.10.2. When $y(x) = x^2$ we saw in Example 1.6.1 that $FS(x)$ was
\[ \frac{\pi^2}{3} - 4(\cos x - \frac{1}{4} \cos(2x) + \frac{1}{9} \cos(3x) + \ldots - (-1)^m \frac{1}{m^2} \cos(mx)) \]
Therefore $c_0 = \frac{\pi^2}{3}$, and for $1 \leq n \leq m$
\[ c_n = 2(-1)^n \frac{1}{n^2} = c_{-n} \]
Now
\[ \|y\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{\pi^4}{5} \]
and Parseval’s Identity says
\[ \frac{\pi^4}{5} \approx \frac{\pi^4}{9} + 8(1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots + \frac{1}{m^4}) \]
Therefore
\[ \frac{\pi^4}{90} \approx 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots + \frac{1}{m^4} \]
The left hand side is approximately 1.082323234. When \( m = 6 \) the right hand side is 1.081123534.

**Example 2.10.3.** Although, strictly speaking, the function \( y \) in Example 1.6.2 is not continuous, Parseval’s Identity can still be applied in this case. Now \( \|y\|^2 = 1 \), and \( a_n = 0 \) for all \( n \). Furthermore,
\[ b_n = \frac{2}{n\pi}(1 - (-1)^n) \]
Therefore for \( 1 \leq n \leq m \)
\[ |c_n| = \frac{1}{n\pi}(1 - (-1)^n) \]
and so
\[ \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \]
The left hand side is approximately 1.233700551. Taking the first 8 terms on the right we get 1.202491020.

Now that we have introduced vector spaces of functions, the main missing ingredient in our discussion of approximations by trigonometric sums is the notion of *convergence* of series.
Chapter 3

Convergence of Trigonometric Series

In Chapter 1 we made some statements about infinite sums of numbers and functions. The discussion was sketchy in parts, especially where we formed sums with infinitely many terms. To make precise statements about infinite sums we need to do some work, but you’ve probably already met some examples of infinite sums already, including some that don’t add up in any sensible way. We’ll start our discussion with one of these.

3.1 Infinite Series

Example 3.1.1. Let $n \in \mathbb{Z}_+$ and consider the sum

$$s_n = 1 + 2 + 3 + \ldots + n$$

If we rewrite the terms in $s_n$ in reverse order we get

$$s_n = n + (n - 1) + (n - 2) + \ldots + 1$$

and adding these two equations gives

$$2s_n = (n + 1) + (n + 1) + (n + 1) + \ldots (n + 1) = n(n + 1)$$

since there are $n$ terms. Therefore

$$s_n = \frac{n(n + 1)}{2}$$

55
which is well-known as the sum of the first $n$ terms in an arithmetic series. If we try to turn this into an infinite sum we will get a number which becomes arbitrarily large as $n \to \infty$. In other words the infinite series (or sum)

$$1 + 2 + 3 + \ldots + n + (n + 1) + \ldots$$

doesn’t converge. \(\Box\)

**Exercise 3.1.1.** The most general arithmetic progression has the form

$$a + d, a + 2d, a + 3d, \ldots a + nd$$

where $a, d \in \mathbb{R}$ and $n \in \mathbb{Z}_+$ are given. Here $d$ is called the common difference. Extend the formula derived in Example 3.1.1 to this more general situation. \(\Box\)

Sometimes infinite sums do add up.

**Example 3.1.2.** Let $r \in \mathbb{R}$ and consider the sum

$$s_n = 1 + r + r^2 + \ldots r^{n-1}$$

where $r$ is called the common ratio. It is easily checked that

$$rs_n = -1 + s_n + r^n$$

so that $(1 - r)s_n = 1 - r^n$ namely

$$s_n = \frac{1 - r^n}{1 - r}$$

which is the well-known formula for the first $n$ terms of a geometric series. To turn this into an infinite sum we should let $n \to \infty$. Three kinds of things can happen, depending on $r$.

- If $|r| > 1$ then $|s_n| \to \infty$ as $n \to \infty$. In this case the infinite series

$$1 + r + r^2 + \ldots + r^{n-1} + r^n + \ldots$$

doesn’t add up to a finite number (the series does not converge). The series also fails to converge when $r = 1$ because then

$$s_n = 1 + 1 + 1 + \ldots 1 = n$$

which also heads off to infinity.
3.1. INFINITE SERIES

- When $r = -1$ it’s easy to check that $s_n$ is either 1 or 0, according as $n$ is odd or even. As $n \to \infty$ the sum $s_n$ doesn’t settle down to any number. Instead it just oscillates between 1 and 0. So the infinite series

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

doesn’t converge either, but its divergence is a little different to that in the previous case.

- In the remaining case, where $|r| < 1$, notice that $r^n \to 0$ as $n \to \infty$. Therefore

$$s_n \to \frac{1}{1 - r}$$

and so we can say that the infinite geometric series

$$1 + r + r^2 + \ldots + r^{n-1} + r^n + \ldots$$

converges to $\frac{1}{1-r}$ in this case.

\[
\square
\]

Exercise 3.1.2. Geometric series frequently arise in simple financial models. Let $a$ be the amount in dollars on an outstanding loan, where the annual interest rate is $100r\%$ (payable at the end of each year). If the interest is not paid, but just gets added onto the loan at the end of each year, give a formula for the amount owing after $n$ years.

It is fairly clear, even from the examples we have considered so far, that a necessary condition for an infinite series to converge is that its $n$th term should go to 0 as $n \to \infty$. It turns out, however, that this necessary condition is not sufficient.

Example 3.1.3. Let $s \in \mathbb{R}$ and consider the infinite series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} + \ldots$$

It turns out that this series coonverges if and only if $s > 1$. In particular the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

diverges.

\[
\square
\]

Exercise 3.1.3. Do some numerical experiments with Mathematica to see how rapidly the partial sums $s_n$ of the harmonic series go to infinity as $n$ increases.
Example 3.1.4. The infinite series
\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^n}{2n+1} + \ldots \]
converges. This can be proved quite easily, using either Taylor series or Fourier series. We’ll discuss this in a moment. \( \square \)

A Taylor series or a Fourier series is an infinite sum of \textit{functions} rather than real numbers. A Taylor series is a \textit{power series} and a Fourier series is a \textit{trigonometric series}. Convergence needs to be looked at in either case.

### 3.1.1 Power Series

A \textit{power series} is an infinite sum of the form
\[ c_0 + c_1 x + c_2 x^2 + \ldots c_n x^n + \ldots \]
where \( x \) is a variable and \( c_0, c_1, c_2, \ldots c_n, \ldots \) are scalars (either real or complex numbers).

Sometimes functions are defined in terms of power series, and interesting properties of functions can be proved using power series representations.

**Exercise 3.1.4.** \( e^x \) can be defined to be
\[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \]
Prove using this power series representation that for any \( n \in \mathbb{Z}_+ \)
\[ \frac{x^n}{e^x} \to 0 \]
as \( x \to \infty \). \( \square \)

The power series representation of \( e^x \) converges for all values of \( x \). Very few power series are as well-behaved as this.

**Exercise 3.1.5.** Show that the power series
\[ 1 + x + x^2 + x^2 + \ldots + x^n + \ldots \]
converges if and only if \( |x| < 1 \). \( \square \)
Power series were once used for estimating values of complicated functions. The basic tool was Taylor’s theorem which says something about the error in approximating a function \( y \) by its Taylor polynomial of a given degree. Now there are better ways of approximating functions, especially using computers and electronic calculators. The algorithms used in these devices are usually more efficient than power series methods.

Power series are still important for theoretical work, and for proving interesting results about series of numbers.

**Example 3.1.5.** For \(|x| < 1\) the power series

\[
1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots
\]

can be summed for each value of \( x \) as a geometric series with common ratio \(-x^2\). So we can write

\[
1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1 + x^2}
\]

Supposing it is all right to integrate both sides of this equation, we have

\[
x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \arctan(x)
\]

Setting \( x = \frac{\pi}{4} \) we obtain

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^n}{2n + 1} + \ldots = \frac{\pi}{4}
\]

The same result was proved in Example 1.6.2 using a trigonometric series.

**Example 3.1.6.** For \(|x| < 1\) we can integrate the power series

\[
\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + \ldots
\]

to get

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots
\]

where the constant of integration is found by substituting \( x = 0 \). The right hand side converges for \( x \in (-1, 1] \) and agrees with \( \ln(1+x) \) over this domain. In particular

\[
\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1} \frac{1}{n} + \ldots
\]
3.1.2 Trigonometric Series

A trigonometric series is an infinite sum of the form

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \]

The Fourier sums in Chapter 1 were trigonometric sums with finitely many terms, chosen to approximate given functions

\[ y : [-\pi, \pi] \to \mathbb{C} \]

Roughly speaking, the more terms we take the better the approximation (it certainly doesn’t get any worse). So the best approximations are got by using trigonometric series. It turns out that trigonometric series in general (and Fourier series in particular) don’t always converge. When a Fourier series does converge it doesn’t always converge to the function \( y \).

These statements are imprecise, because we still need to learn the language of convergence, and to see some theorems about convergence of Fourier series. Convergence can be of several different kinds:

- of series of numbers
- of series of functions (such as power series or trigonometric series)
- of sequences of vectors in a normed vector space

We haven’t even mentioned sequences up to now. They are not the same as series, but a good discussion of convergence of sequences provides us with the necessary tools for dealing with convergence of series.

3.2 Convergence of Sequences

Let \( V \) be a normed vector space. A sequence is a subset

\[ s = \{s_1, s_2, \ldots s_m, \ldots : m \in \mathbb{Z}_+\} \]

of \( V \) indexed by the positive integers.

**Definition 3.2.1.** The sequence \( \{s_m : m \in \mathbb{Z}_+\} \) is said to converge to a limit \( s_\infty \in V \) when, given \( \epsilon > 0 \) there is a number \( M \in \mathbb{Z}_+ \) such that

\[ \|s_m - s_\infty\| < \epsilon \text{ whenever } m \geq M \]
Convergence in the sense of Definition 3.2.1 means that, given a prescribed measure $\epsilon$ of closeness, we can ensure that $s_m$ remains close to $s_\infty$ by taking $m$ larger than some quantity $M$. Usually $M$ depends on $\epsilon$, as we might expect. Although Definition 3.2.1 applies to sequences in any normed vector space, we focus on sequences and series of real numbers in this section. Later on we will use the definition to talk about sequences and series of functions. A sequence can converge to at most one limit, according to the following result.

**Proposition 3.2.1.** Let $s_\infty, \tilde{s}_\infty \in V$ both be limits of the same sequence $s$. Then $\tilde{s}_\infty = s_\infty$.

**Proof:**
Given $\epsilon > 0$ choose $M$ so large that both
\[
\|s_m - s_\infty\| < \epsilon \text{ and } \|s_m - \tilde{s}_\infty\| < \epsilon
\]
whenever $m \geq M$. Then for $m \geq M$ we have
\[
\|s_\infty - \tilde{s}_\infty\| = \|(s_m - s_\infty) - (s_m - \tilde{s}_\infty)\| \leq \|s_m - s_\infty\| + \|s_m - \tilde{s}_\infty\| < \epsilon + \epsilon = 2\epsilon
\]
Therefore $\|s_\infty - \tilde{s}_\infty\| < 2\epsilon$ for any $\epsilon > 0$ and necessarily
\[
\|s_\infty - \tilde{s}_\infty\| = 0
\]
Therefore $s_\infty - \tilde{s}_\infty = 0$ and so $s_\infty = \tilde{s}_\infty$. \(\square\)

When the limit $s_\infty$ of $s$ exists we may also write
\[
\lim_{m \to \infty} s_m = s_\infty
\]

When $s$ converges to some limit the sequence is said to be *convergent*. A sequence that is not convergent is called *divergent*. These are natural extensions to vectors of familiar concepts for real numbers.

**Exercise 3.2.1.** Prove that the sequence of real numbers
\[
s = \left\{ \frac{1}{m} : m \in \mathbb{Z}_+ \right\}
\]
converges to the limit 0. \(\square\)
Exercise 3.2.2. Prove that the sequence
\[ s = \{(-1)^n : n \geq 1\} \]
is divergent.

Exercise 3.2.3. Prove that the sequence of real numbers
\[ s = \{m : m \geq 1\} \]
is divergent.

An important class of examples of sequences comes from series. As in Section 3.1, a series is meant to be a sum with infinitely many terms. When the terms add up to a finite number the series is said to converge. Otherwise the series diverges. More precisely, we have the following definition.

Definition 3.2.2. Let \( a = \{a_n : n \geq 1\} \) be a series of vectors in a normed vector space \( V \). The \( n \)th partial sum associated with \( a \) is
\[ s_n = \sum_{m=1}^{n} a_m = a_1 + a_2 + \ldots + a_n \]
The partial sums \( s_n \) define another sequence
\[ s = \{s_n : n \geq 1\} \]
called the infinite series associated with the sequence \( a \). When \( s \) converges to a limit \( s_\infty \) the series \( \Sigma a_n \) is said to converge also, and then we write
\[ s_\infty = \sum_{n=1}^{\infty} a_n \]
When \( s \) diverges the series \( \Sigma a_n \) is also said to diverge.

A necessary condition for the series \( \Sigma a_n \) associated with a sequence \( a \) to be convergent is that
\[ \lim_{n \to \infty} a_n = 0 \]
This condition is not sufficient for convergence of the series however, as the following example shows.

Example 3.2.1. Let \( a_n = \frac{1}{n} \). Then the associated series is the harmonic series
\[ \sum_{n=1}^{\infty} \frac{1}{n} \]
which is divergent, as noted in Example 3.1.3.
Recall also the following result.

**Proposition 3.2.2.** Fix \( s \in \mathbb{R} \). Then the series

\[
\sum \frac{1}{n^s}
\]

is convergent if and only if \( s > 1 \).

Roughly speaking, if \( a_n \to 0 \) quickly enough the series \( \Sigma a_n \) is likely to be convergent. Another consideration, at least for series of real numbers, is the sign of the terms \( a_n \). A series is said to be alternating when \( a_n \) and \( a_{n+1} \) have opposite signs for every \( n \).

**Proposition 3.2.3.** Let \( a \) be an alternating sequence of real numbers with the property that \( |a_{n+1}| < |a_n| \) for every \( n \geq 1 \). Then the series

\[
\Sigma a_n
\]

is convergent if and only if

\[
\lim_{n \to \infty} a_n = 0
\]

**Proof:**

\[
\Box
\]

**Example 3.2.2.** The alternating series

\[
\Sigma \frac{(-1)^n}{n}
\]

is convergent. In fact the limit is \( \ln 2 \).

\[
\Box
\]

### 3.3 Sequences and Series of Vectors

If we know about convergence of sequences of real numbers, it’s not hard to deal with convergence of sequences of vectors in \( \mathbb{R}^m \) where \( m > 1 \). In the following result, \( n \) is used instead of \( m \) as the dummy variable for indexing elements of sequences.

**Proposition 3.3.1.** Let \( s \) be a sequence of vectors in \( \mathbb{R}^m \). For each \( n = 1, 2, \ldots m \) let \( s^n \) be the sequence of real numbers obtained by taking the \( n \)th coordinates of elements of \( s \). Then \( s \) is convergent if and only if the sequences \( s^n \) of real numbers are all convergent for \( 1 \leq n \leq m \). In such a case

\[
\lim_{n \to \infty} s_n = (\lim_{n \to \infty} s^1_n, \lim_{n \to \infty} s^2_n, \ldots \lim_{n \to \infty} s^m_n)
\]

\[
\Box
\]
Exercise 3.3.1. Prove Proposition 3.3.1

In $\mathbb{R}^m$ (or indeed in any finite-dimensional normed vector space) convergence is independent of the particular norm that is chosen. Usually we take the Euclidean norm on $\mathbb{R}^m$.

Exercise 3.3.2. Let $V = \mathbb{R}^m$ where $m \geq 1$ and let $\| \|$ be any norm on $V$. Prove that a sequence converges with respect to $\| \|$ on $V$ if and only if it converges with respect to the Euclidean norm given by

$$\|(v_1, v_2, \ldots, v_m)\|_E = v_1^2 + v_2^2 + \ldots + v_m^2$$

Exercise 3.3.3. Let $V = \mathbb{C}^m$ where $m \geq 1$ and let $\| \|$ be any norm on $V$. Prove that a sequence converges with respect to the norm $\| \|$ on $V$ if and only if it converges with respect to the Hermitian norm given by

$$\|(v_1, v_2, \ldots, v_m)\|_H = |v_1|^2 + |v_2|^2 + \ldots + |v_m|^2$$

Exercise 3.3.4. State and prove a version of Proposition 3.3.1 for infinite series of vectors, instead of sequences.

A sequence $s$ in any normed vector space $V$ is said to be Cauchy when, given $\epsilon > 0$ there is an integer $M \geq 1$ with the property that

$$\|s_m - s_n\| < \epsilon$$

whenever $m, n \geq M$

In such a case when $m, n$ are large enough $s_m$ and $s_n$ can be guaranteed to be close together. As we might expect, convergent sequences have this property.

Proposition 3.3.2. Let $s$ be a convergent sequence in a normed vector space $V$. Then $s$ is Cauchy.

Proof:

Given $\epsilon > 0$ choose $M$ so large that

$$\|s_m - s_\infty\| < \frac{\epsilon}{2} \text{ for } m \geq M$$

We know we can do this, because $\frac{\epsilon}{2}$ is a positive number which can be used instead of $\epsilon$ in Definition 3.2.1.
Now suppose that \( m, n \geq M \). Then both
\[
\| s_m - s_\infty \| < \frac{\epsilon}{2} \quad \text{and} \quad \| s_n - s_\infty \| < \frac{\epsilon}{2}
\]
whenever both \( m, n \geq M \). In such a case
\[
\| s_m - s_n \| = \| (s_m - s_\infty) - (s_n - s_\infty) \| \leq \| s_m - s_\infty \| + \| s_n - s_\infty \| < \frac{2\epsilon}{2} = \epsilon
\]
This proves the Proposition.

Depending on \( V \) and the norm \( \| \cdot \| \), a Cauchy sequence might or might not be convergent. The normed vector space \( V \) is said to be \textit{complete} when all its Cauchy sequences converge, and \textit{incomplete} otherwise. The reason for inventing real numbers is that \( \mathbb{R} \) is complete, and this property extends to finite-dimensional vector spaces over \( \mathbb{R} \) and \( \mathbb{C} \) as follows.

**Example 3.3.1.** For any \( m \in \mathbb{Z}_+ \), both \( \mathbb{R}^m \) and \( \mathbb{C}^m \) are complete, with respect to any norm.

We can extend \( \mathbb{R}^m \) and \( \mathbb{C}^m \) by considering the set \( \ell_2^\infty \) of all \textit{doubly-infinite sequences} \( c = \{c_m : m \in \mathbb{Z}\} \) of complex numbers for which
\[
\sum_{m \in \mathbb{Z}} |c_m|^2
\]
converges. Strictly speaking this sum is a doubly-infinite series, and we haven’t defined convergence of such. The easiest way to fix this up is to rewrite the series as
\[
|c_0|^2 + \sum_{m \geq 1} (|c_m|^2 + |c_{-m}|^2)
\]
which is the infinite series associated with the sequence
\[
a_1 = |c_0|^2, \ a_2 = |c_1|^2 + |c_{-1}|^2, \ldots \ a_m = |c_m|^2 + |c_{-m}|^2, \ldots
\]

**Exercise 3.3.5.** Show that
- \( \ell_2^\infty \) is a vector space over \( \mathbb{C} \) with respect to the operations of vector addition and scalar multiplication given by
  \[
  (c + d)_m = c_m + d_m \quad \text{and} \quad (r.c)_m = r c_m
  \]
  where \( c, d \in \ell_2^\infty, \ m \in \mathbb{Z} \) and \( r \in \mathbb{C} \)
• $<,>$ given by
  
  $<c,d> = \sum_{m\in\mathbb{Z}} c_m d_m$

  is an inner product on $l^2$

• $l^2$ is complete with respect to the norm associated with $<,>$.

*Hint:* Use the Cauchy-Schwarz inequality for finite sequences. \hfill \square

Convergence of sequences and series of *functions* is a much more delicate issue, and usually more difficult. Outcomes depend on the norm chosen for the vector space $V$.

### 3.4 Sequences and Series of Functions

If $s$ is a sequence of functions $s_n : [-\pi, \pi] \rightarrow \mathbb{R}$ then the first kind of convergence that probably comes to mind is *pointwise convergence*, defined as follows.

**Definition 3.4.1.** The sequence $s$ converges pointwise to a function $s_\infty : [-\pi, \pi] \rightarrow \mathbb{R}$ when, for each $x \in [-\pi, \pi]$, the sequence of real numbers

$$\{s_n(x) : n \geq 1\}$$

converges to $s_\infty(x)$.

*Exercise 3.4.1.* Prove that if $s$ converges pointwise to functions $s_\infty$ and $\tilde{s}_\infty$, then $\tilde{s}_\infty = s_\infty$.

Convergence of a series $\Sigma a_n$ is decided by the convergence of the associated sequence $\{s_n : n \geq 1\}$ of finite sums, as in Section 3.2. This is just as much true for series of functions as for series of real or complex numbers, or for series of vectors in $\mathbb{R}^m$ or $\mathbb{C}^m$.

A *power series* is a series of functions of the form

$$\sum c_n x^{n-1}$$

where $x$ is the independent variable, and the *coefficients* $c_n$ are real or complex numbers. The partial sums of a power series are polynomials, and so power series representations are often used to prove things about relatively complicated functions. Power series are often found from Taylor series expansions. There are some examples in Subsection 3.1.1.
A **trigonometric series** is a series of the form

\[ \frac{a_0}{2} + \sum (a_n \cos(nx) + b_n \sin(nx)) \]

The partial sums of a trigonometric series are trigonometric sums. The trigonometric series approximation of a function \( y \) is found by solving for the coefficients \( a_n, b_n \) using Proposition 1.5. The resulting series is a **Fourier series**. We have seen that Fourier series can be used to find interesting identities involving \( \pi \), but they have other more practical uses, as we see in Chapter ??.

Just as exponential sums are more or less equivalent to trigonometric sums, an **exponential series** can be expressed in terms of a trigonometric series. The general form of an exponential series is

\[ c_0 + \sum (c_n e^{inx} + c_{-n} e^{-inx}) \]

where \( c_n \in \mathbb{C} \) for \( n \in \mathbb{Z} \). Notice that in this case the coefficients \( c_n \) make up a **doubly-infinite sequence** since \( n \) takes all negative as well as positive integer values. As for exponential sums, the coefficients \( c_n \) of an exponential series approximation to a function \( y \) are given by Proposition 1.3.1.

In this book we are mainly interested in convergence of Fourier series and exponential series. The two kinds of series are equivalent, but Fourier series are more useful for doing calculations whereas exponential series are usually better for proving theoretical results. Consequently we use both.

Pointwise convergence of sequences and series of functions (including Fourier series) is interesting and important, but it is not the same as convergence with respect to (say) the uniform norm \( \| \cdot \|_U \) defined next.

### 3.5 The Uniform Norm

Let \( \mathbb{R}B[-\pi, \pi] \) be the vector space of **bounded functions** \( y : [-\pi, \pi] \to \mathbb{R} \), namely \( y \in \mathbb{R}B[-\pi, \pi] \) when there is a bound \( \beta > 0 \) depending on \( y \) with the property that

\[ |y(x)| \leq \beta \text{ for all } x \in [-\pi, \pi] \]

The **uniform norm** \( \| \cdot \|_U \) on \( \mathbb{R}B[-\pi, \pi] \) is given by

\[ \|y\|_U = \text{Sup}\{|y(x)| : x \in [-\pi, \pi]\} \]

Evidently \( \|y\|_U \) is no greater than the constant \( \beta \) referred to above.
Exercise 3.5.1. Verify that $\| \cdot \|_U$ satisfies the conditions for a norm on $B[-\pi, \pi]$. Check that the same holds for the space $C[-\pi, \pi]$ of complex-valued bounded functions.

Convergence with respect to the uniform norm $\| \cdot \|_U$ is called uniform convergence.

Exercise 3.5.2. Prove that if $s$ converges uniformly to $s_\infty$ then $s$ converges pointwise to $s_\infty$.

The difference between uniform convergence and pointwise convergence can be stated as follows.

- $s$ converges uniformly to $s_\infty$ when, given $\epsilon > 0$, there is a number $M$ such that, for all $x \in [-\pi, \pi]$
  \[ |s_m(x) - s_\infty(x)| < \epsilon \]
  whenever $m \geq M$.

- $s$ converges pointwise to $s_\infty$ when, given $\epsilon > 0$ and $x \in [-\pi, \pi]$, there is a number $M$ such that
  \[ |s_m(x) - s_\infty(x)| < \epsilon \]
  whenever $m \geq M$.

With uniform convergence we don’t need to know $x$ before coming up with a suitable number $M$. With pointwise convergence $M$ probably depends on $x$.

The uniform norm is not the only norm that is used on vector spaces of functions, and using a different norm can lead to different statements about convergence.

Example 3.5.1. Let $V = CC[-\pi, \pi]$ and let $\| \cdot \|$ be the norm on $V$ associated with the inner product $\langle \cdot , \cdot \rangle$ where

\[ \langle y, z \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \bar{z}(x) dx \]

For $n \geq 1$ let $s_n(x)$ be 0 for $|x| > \frac{1}{n}$, and $1 - n|x|$ otherwise. Then

- with respect to $\| \cdot \|$, $s = \{ s_n : n \geq 1 \}$ converges to the constant function whose value is 0 on the whole of $[-\pi, \pi]$.
3.5. *THE UNIFORM NORM*

- \( s \) converges pointwise to the function which is 1 at \( x = 0 \) and 0 everywhere else

- \( s \) is divergent with respect to the uniform norm \( \| \cdot \|_U \).


**Exercise 3.5.3.** Prove that the inner product \( \langle \cdot, \cdot \rangle \) defined in Example 3.5.1 satisfies the conditions for an inner product. Do something similar for \( \mathbb{R}C[-\pi, \pi] \).

It can be hard to decide whether a vector space of functions is complete or not. One way to demonstrate incompleteness is to prove that a sequence \( s \) in \( V \) converges to a limit \( s_\infty \) in a larger vector space \( W \), and that \( s_\infty \notin V \).

**Example 3.5.2.** Let \( V = \mathbb{R}C[-\pi, \pi] \) with the uniform norm \( \| \cdot \|_U \). Let \( s_n \in V \) be the \( n \)th partial sum of the exponential function \( e^x \)

\[
s_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}
\]

Then \( s = \{s_n : n \geq 1\} \) can be regarded either as a sequence in \( V \) or as a sequence in the vector subspace \( \mathbb{R}P[x] \) of real polynomial functions with domain \([-\pi, \pi]\). Take the uniform norm on \( \mathbb{R}P[x] \).

As a sequence in \( V \), \( s \) converges to the exponential function, and therefore \( s \) is Cauchy (either in \( V \) or \( \mathbb{R}P[x] \)). As a sequence in \( \mathbb{R}P[x] \), however, \( s \) is divergent. So \( \mathbb{R}P[x] \) is incomplete with respect to the uniform norm.

**Exercise 3.5.4.** Let \( V = \mathbb{R}C[-\pi, \pi] \) with the norm \( \| \cdot \| \) associated with the inner product \( \langle \cdot, \cdot \rangle \) where

\[
\langle y, z \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)z(x)dx
\]

Define \( s_n \) to be the odd function whose value \( s_n(x) \) is 1 for \( x \geq \frac{1}{n} \) and \( \frac{x}{n} \) for \( x \in [0, \frac{1}{n}] \). Prove that \( \{s_n\} \) is Cauchy but diverges as a sequence in \( V \).

Proving completeness of a vector space of functions is sometimes difficult, but in some cases we can establish completeness without much fuss.

**Proposition 3.5.1.** \( B[-\pi, \pi] \) is complete with respect to the uniform norm \( \| \cdot \|_U \).

\(^1\)The norm on \( W \) is supposed to extend the norm on \( V \).
Proof:
Let \( s = \{s_n : n \geq 1\} \) be a Cauchy sequence in \( B[-\pi, \pi] \). Then for any \( x \in [-\pi, \pi] \) the sequence of real numbers
\[
\{s_n(x) : n \geq 1\}
\]
is also Cauchy. Since \( \mathbb{R} \) is complete the sequence of reals converges to some limit. Call this limit \( s_{\infty}(x) \).
The assignment \( x \mapsto s_{\infty}(x) \) defines a function
\[
s_{\infty} : [-\pi, \pi] \to \mathbb{R}
\]
which is a good candidate for the limit of the sequence \( s \). Of course \( s \) converges pointwise to \( s_{\infty} \), but it is another matter to decide whether \( s \) converges uniformly.
Given \( \epsilon > 0 \) choose \( M \) so that \( \|s_m - s_n\|_U < \frac{\epsilon}{3} \) whenever \( m, n \geq M \), according to the definition of a Cauchy sequence. For any \( m \geq M \) we have
\[
|s_m(x) - s_n(x)| < \frac{\epsilon}{3}
\]
for any \( x \in [-\pi, \pi] \) and all \( n \geq M \). Taking limits as \( n \to \infty \) it follows that
\[
|s_m(x) - s_{\infty}(x)| \leq \frac{\epsilon}{3}
\]
for every \( x \). This is the same as saying that
\[
\|s_m - s_{\infty}\|_U \leq \frac{\epsilon}{3} < \epsilon
\]
provided \( m \geq M \). So \( s \) converges uniformly to \( s_{\infty} \).
It remains only to show that \( s_{\infty} \) is a bounded function. This is done as follows. Take \( \epsilon = 1 \) in the definition of convergence. Then because \( s \) converges uniformly to \( s_{\infty} \) we can choose \( M \) so that
\[
\|s_n - s_{\infty}\|_U < 1 \text{ whenever } n \geq M
\]
Consequently
\[
|s_{\infty}(x)| \leq |s_n(x) - s_{\infty}(x)| + |s_n(x)| \leq 1 + |s_n(x)|
\]
for every \( x \in [-\pi, \pi] \).
Fix \( n \geq M \). Then \( s_n \) is a bounded function, namely there is a constant \( \beta > 0 \) with the property that
\[
|s_n(x)| \leq \beta \text{ for every } x \in [-\pi, \pi]
\]
Therefore
\[|s_\infty(x)| \leq 1 + \beta\]
for every \(x \in [-\pi, \pi]\), and \(s_\infty \in B[-\pi, \pi]\) as required. \(\square\)

**Proposition 3.5.2.** Let \(s_\infty\) be a uniform limit of continuous functions \(s_n : [-\pi, \pi] \to \mathbb{R}\). Then \(s_\infty\) is also continuous.

**Proof:**
Given \(\epsilon > 0\) choose \(M\) so that \(\|s_m - s_\infty\| < \frac{\epsilon}{3}\) whenever \(m \geq M\), according to the definition of convergence. Fix \(m \geq M\). Then for every \(x \in [-\pi, \pi]\) we have
\[|s_m(x) - s_\infty(x)| < \frac{\epsilon}{3}\]
Now let \(x \in [-\pi, \pi]\) be given. Because \(s_m\) is continuous at \(x\) we can choose \(\delta > 0\) so that
\[|s_m(x) - s_m(t)| < \frac{\epsilon}{3}\] whenever \(|x - t| < \delta\)
Then \(|s_\infty(x) - s_\infty(t)|\) can be rewritten in the form
\[
|(s_\infty(x) - s_m(x)) + (s_m(x) - s_m(t)) + (s_m(t) - s_\infty(t))| \leq \\
|s_\infty(x) - s_m(x)| + |s_m(x) - s_m(t)| + |s_m(t) - s_\infty(t)| < \\
\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\]
Therefore \(s_\infty\) is continuous at \(x\). Now \(x\) was an arbitrary point in \([-\pi, \pi]\) and so \(s_\infty\) is continuous on the whole of \([-\pi, \pi]\). \(\square\)

**Exercise 3.5.5.** Do the same for the bounded and continuous complex-valued functions \(y\) on \([-\pi, \pi]\). \(\square\)

From Propositions 3.5.1, 3.5.2 it follows that the space \(\mathbb{R}C[-\pi, \pi]\) of real-valued continuous functions on \([-\pi, \pi]\) is also complete with respect to the uniform norm. On the other hand, in Exercise 3.5.4 \(\mathbb{R}C[-\pi, \pi]\) is seen to be incomplete with respect to the norm associated with the inner product.

In fact, any normed vector space \(V\) can be embedded in a larger one that is complete. Up to norm-preserving linear isomorphisms there is a unique smallest complete normed vector space containing \(V\), called the completion of \(V\) with respect to \(\|\cdot\|\). So in some sense incompleteness in normed vector spaces can be fixed by filling in holes.
3.6 The Riemann Integral

Recall that the Riemann integral \( \int_a^b y(x) \, dx \) of a bounded function \( y : [a, b] \to \mathbb{R} \) is defined as a limit (assuming the limit exists) of a family of Riemann sums. The Riemann sums are just sums of signed areas of graphs of piecewise-constant approximations to \( y \). Integrability is a much weaker condition than differentiability (at least for functions defined on closed intervals). Many important functions are integrable, even some which might not seem very well-behaved.

**Example 3.6.1.** The Heaviside function \( H : \mathbb{R} \to \mathbb{R} \) given by

\[ H(x) = 0 \quad \text{for } x < 0 \quad \text{and } H(x) = 1 \quad \text{otherwise} \]

is Riemann-integrable on any closed interval. Of course \( H \) is not continuous at \( x = 0 \), much less differentiable. \( \square \)

A subset \( X \) of \( \mathbb{R} \) is said to be of measure zero when, given \( \epsilon > 0 \), there is a finite union of open intervals containing \( X \), where the lengths of the intervals sum to less than \( \epsilon \). So a set of measure zero is a thin subset of \( \mathbb{R} \). The following result of Henri Lebesgue nails down very precisely the conditions for a function \( y \) to be Riemann-integrable.

**Theorem 3.1.** Let \( y : [a, b] \to \mathbb{R} \) be a bounded function, and let \( X \subset [a, b] \) be the set of all points at which \( y \) is not continuous. Then \( y \) is Riemann-integrable if and only if \( X \) has measure zero. \( \square \)

**Example 3.6.2.** Let \( y : [-\pi, \pi] \to \mathbb{R} \) be the function whose value \( y(x) \) is 1 when \( x \) is a rational number and 0 otherwise. Then for any partition of \( [-\pi, \pi] \) there is a Riemann sum whose value is \( 2\pi \) and another whose value is 0. So no matter how finely we partition \( [-\pi, \pi] \), the Riemann sums don’t converge to any limit. Therefore \( y \) is not Riemann-integrable. \( \square \)

From Theorem 3.1 we obtain

**Corollary 3.2.** Let \( y : [a, b] \to \mathbb{R} \) be Riemann-integrable and let \( g : \mathbb{R} \to \mathbb{R} \) be continuous. Then \( g \circ y : [a, b] \to \mathbb{R} \) is Riemann-integrable. \( \square \)

Let \( R[-\pi, \pi] \) be the space of Riemann-integrable functions \( y : [-\pi, \pi] \to \mathbb{R} \).

**Exercise 3.6.1.** Verify that \( R[-\pi, \pi] \) is a vector space with respect to the usual rules for adding functions, and multiplying a function by a scalar. \( \square \)
From Corollary 3.2 we see that if \( y \in R[−\pi, \pi] \) then \(|y|\) is also Riemann-integrable, and we can define \( \|y\|_1 \) to be
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |y(x)| \, dx
\]
Most of the conditions for \( \| \cdot \|_1 \) to be a norm follow easily from this definition. The exception is the condition that
\[
\|y\|_1 = 0 \implies y = 0
\]
which is false.

**Example 3.6.3.** Let \( y(x) = 0 \) for \( x \neq 0 \) and let \( y(0) = 1 \). Then \( \|y\|_1 = 0 \) but \( y \neq 0 \). \( \square \)

To exclude such examples we can identify functions \( y, z \in R[−\pi, \pi] \) whose difference \( y - z \) satisfies
\[
\|y - z\|_1 = 0
\]
Then \( \| \cdot \|_1 \) becomes a norm on the real vector space \( R[−\pi, \pi] \).

**Example 3.6.4.** takeaway

**Example 3.6.5.** The set \( \mathbb{Q} \) of rational numbers is countable, namely any infinite set of rational numbers can be written as a sequence
\[
q = \{q_1, q_2, \ldots q_n, \ldots \}
\]
Do this for the rational numbers in the interval \([−\pi, \pi]\) and define \( s_n \) to be the Riemann-integrable function whose value \( s_n(x) \) is 1 whenever
\[
x \in \{q_1, q_2, \ldots q_n\}
\]
and 0 otherwise. Let \( s = \{s_n : n \geq 1\} \). Then \( s \) converges pointwise to the non-Riemann-integrable function \( y \) in Example 3.6.2.

On the other hand, according to our rule for identifying functions, each \( s_n \) is identified with the constant function \( 0 \) whose value is 0 everywhere. Naturally the trivial sequence, all of whose entries are \( 0 \), converges to \( 0 \). So \( s \) converges to 0 also. \( \square \)

Notice that \(|y| \in R[−\pi, \pi]\) is a necessary but not sufficient condition for \( y \) to be Riemann-integrable

**Example 3.6.6.** Let \( y(x) = 1 \) for \( x \in \mathbb{Q} \) and \(-1 \) otherwise. Then \( y \) is not integrable on \([−\pi, \pi]\), whereas \(|y| \) is constant. \( \square \)

takeup

\[^2\]Strictly speaking we need the formal notion of an equivalence class to do this.
3.7 Square-Integrable Functions

An alternative route to a norm on spaces of functions is to extend the definition of the inner product defined earlier on \( \mathbb{R}C[-\pi, \pi] \)

\[
<y, z> = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)z(x)dx
\]

If \( y, z \) are Riemann-integrable then so is \( yz \), by the Lebesgue Theorem 3.1. So \( <, > \) defines an inner product on \( R[-\pi, \pi] \), but this time we have to identify functions \( y, z \) for which

\[
<y - z, y - z> = \frac{1}{2\pi} \int_{-\pi}^{\pi} (y(x) - z(x))^2 dx = 0
\]

takeup

**Exercise 3.7.1.** Show that the norm \( \| \|_1 \) is not associated with an inner product on \( R[-\pi, \pi] \).

3.8 The Lebesgue Integral

completeness, an orthonormal family is complete if everything can be written as the limit ...

**Example 3.8.1.**

takeup comparison with Riemann integral

**Exercise 3.8.1.** Let \( R^2[-\pi, \pi] \) be the space of functions \( y : [-\pi, \pi] \rightarrow \mathbb{R} \) whose square \( y^2 \) is Riemann-integrable, namely

\[
\int_{-\pi}^{\pi} y(x)^2 dx \text{ exists}
\]

For \( y, z \in R^2[-\pi, \pi] \) define

\[
<y, z> = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)z(x)dx
\]

Show that this integral \textit{might not exist}, depending on the choices of \( y, z \). Conclude that \( R^2[-\pi, \pi] \) is not a vector space.

**Hint:** Take \( y(x) = 1 \) everywhere and let \( z(x) \) be 1 or \(-1\) according as \( x \in \mathbb{Q} \) or not. Then \( yz \) is not Riemann-integrable. On the other hand \( y, z \in R^2[-\pi, \pi] \).

If \( R^2[-\pi, \pi] \) were a vector space then \( y + z \) and \( y - z \) would be square-integrable. Then \( 4yz = (y + z)^2 - (y - z)^2 \) would be integrable also. 

3.8. THE LEBESGUE INTEGRAL

The Lebesgue integral gives rise to some important examples of infinite-dimensional normed vector spaces.

**Example 3.8.2.** Let $L^1[-\pi, \pi]$ be the space of functions $y : [-\pi, \pi] \to \mathbb{C}$ such that the Lebesgue integral
\[ \int_{-\pi}^{\pi} |y(x)| \, dx \]
exists. It can be shown that $L^1[-\pi, \pi]$ is a vector space over $\mathbb{C}$ and that $\| \|_1$ defined by
\[ \|y\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |y(x)| \, dx \]
nearly defines a norm on $L^1[-\pi, \pi]$.

The reason $\| \|_1$ is not quite a norm is that there are nonzero functions in $L^1[-\pi, \pi]$ for which $\|y\|_1 = 0$. To get over this we identify functions in $L^1[-\pi, \pi]$ which disagree only on sets of measure zero. Takeup after this adjustment it can be shown that $L^1[-\pi, \pi]$ is complete with respect to $\| \|_1$, namely $L^1[-\pi, \pi]$ is a Banach space.

**Example 3.8.3.** A nonzero function $y$ in $L^1[-\pi, \pi]$ for which $\|y\|_1 = 0$. Takeup.

**Exercise 3.8.2.** Show that the norm $\| \|_1$ is not associated with an inner product on $<,>$. Takeup.

**Example 3.8.4.** Let $L^2[-\pi, \pi]$ be the space of functions $y : [-\pi, \pi] \to \mathbb{C}$ which are square-integrable in the sense that the Lebesgue integral
\[ \int_{-\pi}^{\pi} |y(x)|^2 \, dx \]
exists. It turns out that $L^2[-\pi, \pi]$ is a vector space over $\mathbb{C}$ and that $<,>$ given by
\[ <y, z> = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \bar{z}(x) \, dx \]
nearly defines an inner product on $L^2[-\pi, \pi]$.

As with $\| \|_1$ in Example 3.8.2, the difficulty is that there are nonzero functions $y \in L^2[-\pi, \pi]$ for which $<y, y> = 0$. This is got round in the same way as before, namely we identify functions which differ only on sets of measure zero.

Then it turns out that $L^2[-\pi, \pi]$ is complete with respect to the norm $\| \|_2$ associated with the inner product $<,>$, namely $L^2[-\pi, \pi]$ is a Hilbert space. Takeup also some mention of completeness in the other sense.
3.9 Pointwise Convergence of Fourier Series

This Section establishes an interesting and nontrivial result about pointwise convergence of Fourier series and exponential series\(^3\). First we state the famous Riemann-Lebesgue Lemma.

**Lemma 3.9.1.** Let \(y\) be Riemann-integrable on some closed interval \([a, b]\). Then \(y(x)e^{\alpha ix}\) is also integrable and

\[
\lim_{\alpha \to \pm \infty} \int_a^b y(x)e^{\alpha ix} \, dx = 0
\]

**Proof:**
If \(y\) is constant the result follows on integration of \(e^{\alpha ix}\). When \(y\) is not constant approximate it by step functions.

**Corollary 3.3.** If \(y \in R[-\pi, \pi]\) then \(\lim_{n \to \infty} c_n = 0\), where \(c_n\) is the coefficient of \(e^{nix}\) in the exponential series of \(y\).

This is not enough to ensure convergence of Fourier series of Riemann-integrable functions.

**Example 3.9.1.** An integrable function whose Fourier Series diverges.

Let \(y \in R[-\pi, \pi]\) and let \(s_m\) be the \(m\)th partial sum of its exponential series:

\[
s_m(x) = \sum_{n=-m}^{m} c_n e^{nix}
\]

or, in more detail,

\[
s_m(x) = \int_{-\pi}^{\pi} y(t)D_m(x-t) \, dt = \int_{x-\pi}^{x+\pi} y(t)D_m(x-t) \, dt = \int_{-\pi}^{\pi} y(x-t)D_m(t) \, dt
\]

where

- \(D_m(t) = \frac{1}{2\pi} \sum_{n=-m}^{m} e^{n\pi t}\) and
- \(y\) is extended periodically outside the domain \([-\pi, \pi)\)

\(^3\)The two are equivalent, as for finite sums.
Summing geometric series, \(2\pi D_m(t)\) takes the form
\[
\frac{1 - e^{(m+1)it}}{1 - e^{it}} + \frac{1 - e^{-(m+1)it}}{1 - e^{-it}} - 1 =
\]
\[
\frac{e^{it/2} - e^{-it/2} + e^{(m+1/2)it} - e^{-(m+1/2)it}}{e^{it/2} - e^{-it/2}} - 1 =
\]
\[
\frac{e^{(m+1/2)it} - e^{-(m+1/2)it}}{e^{it/2} - e^{-it/2}} = \frac{\sin((m + 1/2)t)}{\sin(t/2)}
\]

at least if \(t\) is not an integral multiple of \(2\pi\). This proves

**Lemma 3.9.2.** For \(t\) not an integer multiple of \(2\pi\), \(D_m(t)\) can be written in the form
\[
\frac{1}{2\pi} \sin((m + 1/2)t) \quad \frac{\sin(t/2)}{\sin(t/2)}
\]

\(\square\)

**Exercise 3.9.1.** Determine \(D_m(t)\) when \(t = 2k\pi\) and \(k \in \mathbb{Z}\). \(\square\)

Because \(D_m\) is an even function, \(s_m(x)\) can be written as
\[
\int_0^\pi (y(x + t) + y(x - t))D_m(t)dt =
\]
\[
2 \int_0^{\pi/2} (y(x + 2t) + y(x - 2t))D_m(2t)dt =
\]
\[
\frac{1}{\pi} \int_0^{\pi/2} (y(x + 2t) + y(x - 2t)) \frac{\sin((2m + 1)t)}{\sin t} dt
\]

**Exercise 3.9.2.** Define \(f : [0, \frac{\pi}{2}] \to \mathbb{R}\) by \(f(0) = 0\) and
\[
f(t) = \frac{1}{t} - \frac{1}{\sin t} \quad \text{for} \quad t > 0
\]

Show that \(f\) is continuous. \(\square\)

From Exercise 3.9.2 and the Riemann-Lebesgue Lemma, \(\lim_{m \to \infty} s_m\) is
\[
\frac{1}{\pi} \lim_{m \to \infty} \int_0^{\pi/2} (y(x + 2t) + y(x - 2t)) \frac{\sin((2m + 1)t)}{t} dt
\]
Another application of Lemma 3.9.1 gives instead
\[
\frac{1}{\pi} \lim_{m \to \infty} \int_0^\delta g(t) \frac{\sin((2m + 1)t)}{t} dt
\]
for any \( \delta \in (0, \frac{\pi}{2}] \) where
\[
g(t) = y(x + 2t) + y(x - 2t)
\]
and \( x \) is fixed.

**Exercise 3.9.3.** Show that the improper integral
\[
\int_0^\infty \frac{\sin t}{t} dt
\]
exists.

*Hint:* Use integration by parts to replace the integrand by something which converges more rapidly to 0 as \( t \to \infty \).

**Lemma 3.9.3.** Let \( g \) be continuously differentiable on \([0, \delta)\) namely that, for some \( \tilde{\delta} > 0 \), \( g \) extends to a function
\[
\tilde{g} : (-\tilde{\delta}, \delta) \to \mathbb{C}
\]
where \( \tilde{g}'(t) \) exists and varies continuously for \( t \in (-\tilde{\delta}, \delta) \). Then
\[
\lim_{m \to \infty} \int_0^\delta g(t) \frac{\sin((2m + 1)t)}{t} dt = g(0) \kappa
\]
where
\[
\kappa = \int_0^\infty \frac{\sin t}{t} dt
\]

**Proof:**

By Taylor’s theorem, \( \tilde{g}(t) = g(0) + t\tilde{h}(t) \), where \( \tilde{h} \) is continuous on some open interval containing \( 0 \). Then
\[
\int_0^\delta g(t) \frac{\sin((2m + 1)t)}{t} dt = g(0) \int_0^\delta \frac{\sin((2m + 1)t)}{t} dt + \int_0^\delta h(t) \sin((2m + 1)t) dt
\]
Taking limits as \( m \to \infty \) we are left with
\[
g(0) \lim_{m \to \infty} \int_0^\delta \frac{\sin((2m + 1)t)}{t} dt
\]
on the right hand side, by Lemma 3.9.1 again. A change of variables in the integral gives
\[ g(0) \int_0^\infty \frac{\sin(u)}{u} du \]

Before applying Lemma 3.9.3 it’s necessary to note that the function \( g \) which we manufactured out of \( y \) doesn’t necessarily satisfy the hypotheses of the lemma. The hypotheses will be satisfied if either \( y \) is continuously differentiable on some open interval containing \( x \), or if both

- \( x \) is an isolated point of finite jump discontinuity for \( y \) which is continuously differentiable nearby, and
- \( y \) is right-continuously differentiable at \( x \) after replacing \( y(x) \) by \( y(x+) = \lim_{h \to 0^+} y(x+h) \), and
- \( y \) is left-continuously differentiable at \( x \) after replacing \( y(x) \) by \( y(x-) = \lim_{h \to 0^-} y(x+h) \)

When these conditions on \( y \) are satisfied at every \( x \in (-\pi, \pi) \) call \( y \) amenable. We have proved

**Proposition 3.9.1.** Let \( y \) be amenable\(^4\). Then
\[ \lim_{m \to \infty} s_m = \frac{\kappa}{\pi}(y(x+) + y(x-)) \]

Now \( \kappa \) is independent of \( y \), provided only that \( y : [-\pi, \pi] \to \mathbb{C} \) is amenable.

**Example 3.9.2.** Take \( y(x) = 1 \) for all \( x \in [-\pi, \pi] \). Then
\[ c_n = \frac{1}{2\pi} \pi e^{-ni} dx = \delta_n^0 \]

Then the Fourier series \( \lim_{m \to \infty} s_m \) is identically 1. Comparing with Proposition 3.9.1 (and evaluating at some \( x \in (-\pi, \pi) \) we obtain
\[ 1 = 2\left(\frac{\kappa}{\pi}\right) \]

Consequently
\[ \int_0^\infty \frac{\sin t}{t} dt = \kappa = \frac{\pi}{2} \]

\(^4\)These conditions on \( y \) can be substantially weakened to permit functions of bounded variation.
Putting this together, we have a proof of the Dirichlet-Jordan Theorem, at least in the case where \( y \) is amenable. The Dirichlet-Jordan Theorem is actually true when \( y \) is of bounded variation which is more general than amenability.

**Theorem 3.4.** Let \( y : [-\pi, \pi] \rightarrow \mathbb{C} \) be amenable. Then the Fourier series

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))
\]

and the associated exponential series

\[
\sum_{n=-\infty}^{\infty} c_n e^{inx}
\]

of \( y \) both converge pointwise to

\[
\frac{y(x+) + y(x-)}{2}
\]

\[\square\]

In particular, if \( y \) is continuously differentiable on some open interval containing \( x \in (-\pi, \pi) \) then the Fourier and exponential series of \( y \) at \( x \) converge to \( y(x) \).

### 3.10 The Gibbs Phenomenon

One of the remarkable things about Fourier series is that they can be used to represent discontinuous functions\(^5\) (as well as continuous functions) as trigonometric series. Of course we have to take care in what we say about such representations. Even in simple cases the Fourier series might not converge pointwise to \( y \).

**Example 3.10.1.** Let \( H : [-\pi, \pi] \rightarrow \mathbb{R} \) be the Heaviside function namely \( H(x) \) is 1 for \( x \geq 0 \) and 0 otherwise. Then \( H \) is amenable in the sense of Section 3.9 and therefore its Fourier series converges to \( 0, \frac{1}{2} \) or 1 according as \( x < 0, x = 0 \) or \( x > 0 \). So the Fourier series does not converge pointwise to \( H \) exactly but it does converge to a rather similar discontinuous function. \[\square\]

\(^5\)It is not true, however, that all functions can be represented by Fourier series.
It’s surprising that discontinuous functions can be well-approximated by a series of continuous functions, and there’s a price to be paid. The first thing we need to point out is that if $y$ is discontinuous anywhere in $(-\pi, \pi)$ then the Fourier series cannot possibly converge \textit{uniformly} to $y$. The reason for this is Proposition 3.5.2 which says that a uniform limit of continuous functions has to be continuous. If $y$ isn’t continuous it cannot be a uniform limit of its trigonometric sums $s_n$.

Because the Fourier series does not converge uniformly it follows from the definition of the uniform norm $\| \cdot \|_U$ that, for any $s_\infty : [-\pi, \pi] \to \infty$ (and in particular for the pointwise limit of the Fourier series), the following statement holds.

There is a positive constant $\epsilon \in \mathbb{R}$ such that, given any $M \geq 1$ there is some $x \in (-\pi, \pi)$ with the property that

$$|y_m(x) - s_\infty(x)| \geq \epsilon$$

for some $m \geq M$.

If $x$ is fixed then $s_m(x)$ can be made close to $s_\infty(x)$ just by making sure $m$ is large. How large depends on $x$, and no matter how large $m$ is there will always be some $x$ for which the approximation of $s_\infty(x)$ by $s_m(x)$ is not very good. This state of affairs is called the \textit{Gibbs phenomenon}. It’s not as puzzling as it sounds at first, especially when you consider the difficulty of approximating the Heaviside function by continuous functions in a neighbourhood of 0. It also helps to do some experimentation with a mathematical software package.

**Example 3.10.2.** Open \textit{Mathematica} and load the DiracDelta package by entering

```mathematica
<<Calculus'DiracDelta'
```

Plot the graph of the Heaviside Function with

```mathematica
Plot[UnitStep[x],{x,-Pi,Pi}]
```

Next load the FourierTransforms package by entering

```mathematica
<<Calculus'FourierTransform'
```

and plot the graph of the (say) 5th partial sum of the Heaviside Function.

```mathematica
z=FourierExpSeries[UnitStep[x],{x,-Pi,Pi},5];
Plot[z,{x,-1,1}]
```

Try replacing the 5 by larger numbers, and see what happens near $w = 0, -\pi, \pi$. 
\qed
3.11 Parseval’s Identity

In Section 2.10 Parseval’s identity is an interesting statement about the coefficients $c_m$ of an exponential sum. This extends naturally to a statement about exponential series, and applies to a much larger class of functions. When we look at Parseval’s identity from this point of view its statement actually takes on a simpler form, in terms of a linear transformation $E$ between infinite dimensional normed vector spaces. We have met these vector spaces already.

- The domain of $E$ is the space $L^2[-\pi, \pi]$ of functions $y : [-\pi, \pi] \to \mathbb{C}$ which are square-integrable, namely the Lebesgue integral

$$\int_{-\pi}^\pi |y(x)|^2 dx$$

exists. We have not been very precise about the Lebesgue integral, although a number of its properties have been mentioned. In particular, when $y, z \in L^2[-\pi, \pi]$ the integral

$$<y, z> = \frac{1}{2\pi} \int_{-\pi}^\pi y(x)\bar{z}(x)dx$$

exists, and $<\ , \ >$ is nearly an inner product on $L^2[-\pi, \pi]$. The reason $<\ , \ >$ is not quite an inner product is that there are nonzero functions $y \in L^2[-\pi, \pi]$ for which $<y, y> \neq 0$. In order to get round this we identify functions in $L^2[-\pi, \pi]$ which disagree only on sets of measure zero.

- The range of $E$ is the vector space $\ell^2$ of doubly-infinite sequences $c = \{c_m : m \in \mathbb{Z}\}$ of complex numbers, considered in Section 3.3. Recall that $c$ satisfies the convergence condition

$$\sum_{m \in \mathbb{Z}} |c_m|^2 < \infty$$

and that there is an inner product $<\ , \ >$ on $\ell^2$ given by

$$<c, d> = \sum_{m \in \mathbb{Z}} c_m \bar{d}_m$$

Furthermore $\ell^2$ is complete with respect to the norm associated with this inner product: $\ell^2$ is a Hilbert space.
The definition of $E : L^2[-\pi, \pi]$ is now strongly suggested by our notation, namely $E$ maps $y \in L^2[-\pi, \pi]$ to its sequence

$$c = \{c_m : m \in \mathbb{Z}\}$$

of coefficients in its exponential series

$$\Sigma c_m e^{i mx}$$

**Exercise 3.11.1.** Show that $E$ is a linear transformation, namely that

- $E(y + z) = E(y) + E(z)$ and
- $E(ry) = rE(y)$

where $y, z \in L^2[-\pi, \pi]$ and $r \in \mathbb{C}$.

Parseval’s identity now takes the form

**Proposition 3.11.1.** The linear transformation $E$ is an isometry in the sense that

$$<E(y), E(z)> = <y, z>$$

**Exercise 3.11.2.** Prove Proposition 3.11.1.

So the linear transformation $E$ respects the inner products on $L^2[-\pi, \pi]$ and $\ell^2$. It happens that $E$ is also a linear isomorphism, so that the normed vector spaces $L^2[-\pi, \pi]$ and $\ell^2$ have many properties in common. For instance it is reasonably straightforward to show that $\ell^2$ is complete. It follows that $L^2[-\pi, \pi]$ is complete also. For applications of Parseval’s identity see Section 2.10.

### 3.12 Convolutions of Functions

Let $y, z \in L^1[-\pi, \pi]$. Their convolution $y \ast z \in L^1[-\pi, \pi]$ is given by the formula

$$(y \ast z)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x - u)z(u)du$$

Some comments about the definition:
In order that the integrand should exist for all \((x, u) \in [−\pi, \pi] \times [−\pi, \pi]\)
it is assumed that \(y\) is extended to a periodic function of period \(2\pi\) defined over the whole of \(\mathbb{R}\).

The existence of the integral for almost every value of \(x\), and the fact that \(y \ast z \in L^1[−\pi, \pi]\) is part of the theory of the Lebesgue integral.

Convolution is a complicated operation but it occurs a lot in applications.

**Example 3.12.1.** Let \(Y\) and \(Z\) be continuous random variables taking values in the unit circle \(S^1 = \{(\cos \theta, \sin \theta) : \theta \in [−\pi, \pi]\}\).

Values of \(Y\) and \(Z\) are conveniently given by angles \(\theta_Y\) and \(\theta_Z\) respectively. Probabilities may be given by probability density functions

\[y, z : [−\pi, \pi) \rightarrow \mathbb{R}\]

where the probability that \(\theta_Y\) lies in \([a, b] \subset [−\pi, \pi]\) is

\[\int_a^b y(\theta) d\theta\]

and similarly for \(\theta_Z\). The probability densities should be everywhere non-negative and should integrate to 1

\[\int_{−\pi}^{\pi} y(\theta) d\theta = \int_{−\pi}^{\pi} z(\theta) d\theta = 1\]

There is some possible ambiguity about the angles \(\theta_Y, \theta_Z\) corresponding to values of \(Y\) and \(Z\). So extend \(y\) and \(z\) periodically to functions defined on the whole of \(\mathbb{R}\).

Now of \(S^1\) as the unit circle in the complex plane \(\mathbb{C}\), namely the set of complex numbers of length 1. Complex multiplication of two such numbers gives another number on \(S^1\), and so we can define the product \(YZ\) of the random variables \(Y\) and \(Z\). Of course

\[\theta_{YZ} = \theta_Y + \theta_Z \mod 2\pi\]

Then the probability density function of \(YZ\) turns out to be

\[2\pi y \ast z\]
Example 3.12.1 makes it more or less inevitable that the integral of $y \ast z$ is nicely related to $\|y\|_1$ and $\|z\|_1$, at least when $y$ and $z$ are everywhere non-negative. More precisely, we have

**Proposition 3.12.1.** Let $y, z \in L^1[-\pi, \pi]$ be functions which are everywhere non-negative. Then

$$\|y \ast z\|_1 = \|y\|_1 \|z\|_1$$

**Proof:**

Writing out $\|y \ast z\|_1$ we obtain

$$(\frac{1}{2\pi})^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(x - u)z(u)dudx$$

since $yz$ is everywhere non-negative. Assuming that it is permissible to reverse the order of integration we get instead

$$(\frac{1}{2\pi})^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(x - u)z(u)dxdzu$$

and the change of variable $v = x - u$ gives

$$(\frac{1}{2\pi})^2 \int_{-\pi}^{\pi} \int_{-\pi}^{-u} y(v)z(u)dvdudu = (\frac{1}{2\pi})^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(v)z(u)dvdudu$$

where the change in limits is justified because $y$ is periodic of period $2\pi$. The variables $u, v$ are now decoupled for the purposes of integration and so we can write

$$\|y \ast z\|_1 = (\frac{1}{2\pi})^2 \int_{-\pi}^{\pi} y(v)dv \int_{-\pi}^{\pi} z(u)du = \|y\|_1 \|z\|_1$$

since $y, z$ are everywhere non-negative. \qed

Convolution has other nice properties too, including the following which you should be able to prove.

**Proposition 3.12.2.** Let $y, z, w \in L^1[-\pi, \pi]$. Then

1. $y \ast z = z \ast y$

2. $(y + z) \ast w = y \ast w + z \ast w$
Finally we have an interesting relationship between convolutions and Fourier series, which can be conveniently stated in terms of exponential series of the functions $y, z, y \ast z$.

Suppose that the exponential series of $y$ and $z$ are

$$
\sum_{m \in \mathbb{Z}} c_m e^{imx} \quad \text{and} \quad \sum_{m \in \mathbb{Z}} d_m e^{imx}
$$

Writing

$$E(y) = \{c_m : m \in \mathbb{Z}\} \quad \text{and} \quad E(z) = \{d_m : m \in \mathbb{Z}\}$$

we have

**Proposition 3.12.3.** $E(y \ast z) = \{c_md_m : m \in \mathbb{Z}\}$

**Proof:**

The $m$th coefficient of the exponential series of $y \ast z$ is

$$
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(x-u)z(u)e^{-imx}du dx =
$$

$$
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(x-u)e^{-im(x-u)}z(u)e^{-imiu}du dx
$$

Reversing the order of integration, we get

$$
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(x-u)e^{-im(x-u)}dx z(u)e^{-imiu}du =
$$

$$
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi-u} y(v)e^{-imv}dv z(u)e^{-imiu}du
$$

after the change of variables $v = x - u$. Now $y(v)e^{-imv}$ is periodic of period $2\pi$ and so we can change the limits of integration in the inner integral to obtain

$$
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} y(v)e^{-imv}dv z(u)e^{-imiu}du
$$

The integrals can now be decoupled, giving

$$
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} y(v)e^{-imv}dv \int_{-\pi}^{\pi} z(u)e^{-imiu}du = \|y\|_1 \|z\|_1
$$

So $E$ converts the complicated operation of convolution into the simpler operation of multiplication of elements of doubly-infinite sequences. Sometimes it is necessary to solve convolution equations where unknown functions appear in convolution with other functions. Applying $E$ can then give an algebraic equation for the exponential coefficients, where convolution is replaced by multiplication. Usually the algebraic equation is easier to solve.
3.13 Convolutions of Sequences

If \( y, z \in L^2[-\pi, \pi] \) then their product \( yz \) is in \( L^1[-\pi, \pi] \). As before, write
\[
E(y) = \{ c_m : m \in \mathbb{Z} \} \quad \text{and} \quad E(z) = \{ d_m : m \in \mathbb{Z} \}
\]
for the exponential coefficients of \( y \) and \( z \).

**Proposition 3.13.1.** The \( m \)th exponential coefficient of \( yz \) is
\[
\Sigma_{n \in \mathbb{Z}} c_{m-n} d_n
\]

**Proof:**
Write
\[
y(x) = \Sigma_{p \in \mathbb{Z}} c_p e^{p i x} \quad \text{and} \quad z(x) = \Sigma_{n \in \mathbb{Z}} d_n e^{n i x}
\]
Then \( y(x)z(x) \) is
\[
\Sigma_{m \in \mathbb{Z}} \Sigma_{p+n=m} c_p d_n e^{m i x}
\]

Call the sequence
\[
\{ \Sigma_{n \in \mathbb{Z}} c_{m-n} d_n : m \in \mathbb{Z} \}
\]
the convolution \( c * d \) of the sequences \( c = \{ c_m : m \in \mathbb{Z} \} \) and \( d = \{ d_m : m \in \mathbb{Z} \} \). Then \( E^{-1} \) converts convolutions of sequences in \( \uparrow^2 \) into the products of corresponding functions.

**Exercise 3.13.1.** Let \( c, d, e \in \uparrow^2 \). Prove directly (without using Proposition 3.13.1) that

- \( c * d = d * c \)
- \( (c + d) * e = c * e + d * e \)

Reprove these facts using Proposition 3.13.1.

The main application of Fourier series to problems in physics and engineering is by way of *partial differential equations*. It’s harder to solve partial differential equations (PDEs) than ordinary differential equations (ODEs). If you know how to solve second order linear ODEs, that’s probably enough for our purposes. If you feel a bit rusty you might prefer to take a look at [?] before moving on to Chapter ???. In [?] we say what ODEs are, how to solve them in special cases, and how ODEs enter into physics and engineering. In Chapters 4, 5 we do the same for PDEs.
Chapter 4

Partial Derivatives and Applications

Here we look at some applications of importance in physics and engineering

- vibrations of strings and membranes
- heat flow
- fluid mechanics
- electrostatics and electromagnetics

In each case physical laws are stated in terms of partial derivatives. The statements take the form of partial differential equations (PDEs). In Chapter 5 we focus on some simple examples of PDEs, namely the 1-dimensional heat equation and the 1-dimensional wave equation. These can be used to model the flow of heat along a uniform rod, and the vibrations of a uniform elastic string respectively.

4.1 Heat Flow and Laplacians

Let $R$ be a uniform rod placed along the $x$-axis. If $y(x, t)$ is the heat per unit length at location $x$ and time $t$, then the temperature of $R$ is the function given by

$$ T(x, t) = \frac{1}{\rho \sigma} \frac{\partial y}{\partial x} $$
where $\rho$ is the mass per unit length, and $\sigma \in \mathbb{R}$ is another constant called the specific heat. Fourier’s law for the flow of heat says

$$\frac{\partial y}{\partial t} = \kappa \frac{\partial T}{\partial x}$$

where $\kappa \in \mathbb{R}$ is another constant. Then we obtain the 1-dimensional heat equation

$$\frac{\partial y}{\partial t} = c \frac{\partial^2 y}{\partial x^2}$$

(4.1)

where $c = \frac{\kappa}{\rho \sigma}$.

Similar statements can be made concerning flow of heat in 2-dimensional and 3-dimensional media. The corresponding wave equations have the form

$$\frac{\partial y}{\partial t} = c \Delta y$$

(4.2)

where in the case of a 3-dimensional medium the Laplacian $\Delta$ is

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

4.2 Vibrations and Laplacians

Let $S$ be an elastic string stretched between two points distance $L$ apart. We want to know how $S$ moves when it is permitted to vibrate from a given initial position, for example when the string is plucked. Let $S$ be uniform of mass $\rho$ per unit length.

If the string is light and sufficiently stretched, the dominant force is the tension $T$ in $S$, which we assume to be constant along the string. Without loss $S$ is stretched horizontally between $(0, 0)$ and $(L, 0)$. Then its configuration is described by a function

$$y : [0, L] \times \mathbb{R} \to \mathbb{R}$$

where $(x, y(x, t))$ is the position of a point on $S$ at time $t$.

For a short segment of $S$ of length$^1$ $\delta x$ starting at $(x, y) \in \mathbb{R}^2$, the initial angle $\theta \in [-\pi, \pi)$ of inclination at $(x, y(x, t))$ is given by

$$\tan \theta = \left. \frac{\partial y}{\partial x} \right|_{(x,t)}$$

$^1$Here $\delta x$ is a small increment in $x$, $\delta \theta$ the corresponding increment in $\theta$ and so on.
4.2. VIBRATIONS AND LAPLACIANS

The total horizontal and vertical forces on the segment are

\[-T \cos \theta + T \cos(\theta + \delta \theta) = T \sin \theta \delta \theta + O(\delta \theta)^2\]

and

\[-T \sin \theta + T \sin(\theta + \delta \theta) = T \cos \theta \delta \theta\]

respectively. For \( \theta \approx 0 \) these forces are approximately 0 and \( T \delta \theta \) to first order in \( \theta \) and \( \delta \theta \). So there is no appreciable sideways movement, and the vertical motion satisfies

\[\rho \delta x \frac{\partial^2 y}{\partial t^2} = T \delta \theta\]

according to Newton’s second law, where \( \delta \theta \). Now

\[\sec^2 \theta \delta \theta \approx \frac{\partial^2 y}{\partial x^2} \delta x\]

and therefore, passing to the limit as \( \delta x \to 0 \),

\[\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}\] (4.3)

at least when \( \theta \approx 0 \). Write

\[\frac{T}{\rho} = c^2\]

Then (4.4) is the 1-dimensional wave equation

\[\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\] (4.4)

A similar analysis can be carried out for a vibrating membrane (such as the surface of a drum). This leads to the 2-dimensional wave-equation

\[\frac{\partial^2 y}{\partial t^2} = c^2 \Delta y\] (4.5)

for the height \( y(x, t) \) of the membrane above a point \( x \in \mathbb{R}^2 \) in a plane containing the boundary. Here \( \Delta \) is the 2-dimensional Laplacian

\[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\]

in two space variables \( x_1, x_2 \) where \( x = (x_1, x_2) \in \mathbb{R}^2 \).
4.3 Gradient and Curl

A path-connected subset $U$ of $\mathbb{R}^n$ is said to be *simply-connected* when any closed loop in $U$ can be continuously contracted down to a constant map.

**Exercise 4.3.1.** Prove that the unit sphere $S^2$ in $\mathbb{R}^3$ is simply connected. What would you say about a solid torus in $\mathbb{R}^3$?

**Definition 4.3.1.** Let $X$ be some quantity depending smoothly on $x \in U$ where $U$ is an open subset of $\mathbb{R}^n$. The *gradient* of $X$ is defined as

$$\nabla X = \begin{bmatrix} \frac{\partial X}{\partial x_1} \\ \frac{\partial X}{\partial x_2} \\ \vdots \\ \frac{\partial X}{\partial x_n} \end{bmatrix}$$

**Exercise 4.3.2.** Let $f : U \to \mathbb{R}$ be a smooth function where $U$ is open in $\mathbb{R}^n$. Let $x \in U$ and suppose that $\nabla f(x) \neq 0$. Show that

1. the unit vector in the direction of $\nabla f(x)$ is the direction of greatest increase of $f$ at $x$
2. the unit vector in the direction of $-\nabla f(x)$ is the direction of greatest decrease of $f$ at $x$
3. if $\omega : (-\delta, \delta) \to \mathbb{R}^n$ is a smooth curve with $\omega(0) = x$ and satisfying $f(\omega(t)) = f(x)$ for all $t \in (-\delta, \delta)$ then $\nabla f(x)$ is orthogonal to $\dot{\omega}(0)$

**Definition 4.3.2.** If $\psi : U \to \mathbb{R}$ is a smooth real-valued function then its negative\(^2\) gradient $-\nabla \psi$ is a smooth vector field $V$ on $U$, namely a smooth assignment of a vector in $\mathbb{R}^n$ to every element of $U$. Then $\psi$ is said to be a *potential function* of $V$.

When a smooth vector field has a potential function $V$ on an open subset $U$ of $\mathbb{R}^n$ has a potential function, $V$ is said to be *conservative*. Not all vector fields are conservative and it is important to be able to decide which are.

\(^2\)The - sign is a convention that we follow, but very little would change if it was a + sign instead.
Exercise 4.3.3. Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be smooth. Prove that $V$ is conservative.

Exercise 4.3.4. Let $U$ be an open subset of $\mathbb{R}^n$ and, for some $\delta > 0$, let $\omega : (-\delta, \delta) \rightarrow U$ be a smooth curve. Let $V$ be a conservative vector field on $U$ with potential function $\phi$. Suppose that the Euclidean inner product

$$< V(\omega(t)), \omega(t) >$$

is positive for every $t \in (-\delta, \delta)$. Show that

1. the composite $\phi \circ \omega : (-\delta, \delta) \rightarrow \mathbb{R}$ is a decreasing function of $t$, and that
2. $\omega(s) \neq \omega(t)$ for any $-\delta < s < t < \delta$.

Exercise 4.3.5. Prove that $V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$V(x_1, x_2) = (-x_2, x_1)$$

is not conservative.

Hint: Consider $\omega : \mathbb{R} \rightarrow \mathbb{R}^2$ where

$$\omega(t) = (\cos t, \sin t)$$

Proposition 4.3.1. If a smooth vector field $V$ on an open subset $U$ of $\mathbb{R}^3$ is conservative then its curl

$$\nabla \times V = \begin{bmatrix} \frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3} \\ \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1} \\ \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2} \end{bmatrix}$$

satisfies

$$\nabla \times V = 0 \quad (4.6)$$

where the right hand side is the trivial vector field whose value is everywhere 0.

Exercise 4.3.6. Prove Proposition 4.6.
Notice that

- $\nabla \times$ operates on vector fields $V$ whose domains are open subsets of $\mathbb{R}^3$,
- $\nabla$ operates on real-valued functions whose domains are open subsets of $\mathbb{R}^n$ for any $n \in \mathbb{Z}_+$

and in either case the result is a vector field.

A vector field $V$ on an open subset $U$ of $\mathbb{R}^3$ is said to be irrotational when $\nabla \times V = 0$. If $V$ is conservative then it must be irrotational and (less obviously) the converse holds when $U$ is simply-connected.

**Proposition 4.3.2.** Let $U$ be a simply-connected open subset of $\mathbb{R}^3$. Then $V$ is conservative if and only if it is irrotational. \(\square\)

When $U$ is not simply-connected an irrotational vector field on $U$ may fail to be conservative.

**Example 4.3.1.** Define $V : \mathbb{R}^3 \setminus \{(0,0)\} \times \mathbb{R} \to \mathbb{R}^3$ to be the vector field given by

$$V(x) = \frac{(-x_2, x_1, 0)}{(x_1^2 + x_2^2)\alpha}$$

where $\alpha \in \mathbb{R}$ is constant. The first two components of $\nabla \times V$ are evidently 0, and the third is

$$\frac{2(1 - \alpha)}{(x_1^2 + x_2^2)\alpha}$$

So $V$ is irrotational if and only if $\alpha = 1$.

Choose $\epsilon > 0$. Then, regardless of $\alpha$, $V$ is everywhere tangent to the smooth curve $\omega : (-\pi - \epsilon, \pi + \epsilon) \to \mathbb{R}^3$ given by

$$\omega(\theta) = (\cos \theta, \sin \theta, 0)$$

Now $\omega(-\pi) = \omega(\pi)$. So, by Exercise 4.3.4 (2), $V$ is not conservative. \(\square\)

### 4.4 Stokes’ Theorem and Divergence

The *divergence* of a vector field $V$ on some open subset $U$ of $\mathbb{R}^3$ is defined to be

$$\text{div}(V) \equiv <\nabla, V> := \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}$$

So the divergence of a vector field is a scalar-valued function\(^3\).

\(^3\)On the other hand, the curl $\nabla \times V$ of $V$ is a vector field.
Exercise 4.4.1. Verify that if $V, W$ are vector fields on $U$ and $a, b \in \mathbb{R}$ then
\[< \nabla, aV + bW > = a < \nabla, V > + b < \nabla, W > \]

Exercise 4.4.2. Verify that $< \nabla, \nabla \times V > = 0$. 

Unlike $\nabla$ (which it resembles in appearance) $\Delta$ transforms a scalar-valued function $\phi$ into another function. The result of applying $\nabla$ however is the gradient vector field $\nabla \phi$. If we take the divergence of the gradient this gives the Laplacian
\[< \nabla, \nabla \phi > = \Delta \phi \]

Exercise 4.4.3. Verify that
\[\nabla \times (\nabla \times V) = \nabla < \nabla, V > - \Delta V \]
where the Laplacian $\Delta$ is applied separately to each component of the vector field $V$. 

The divergence figures in Stokes’ theorem in $\mathbb{R}^3$ which says that, for a closed smooth region $R$ in $U$, the surface integral of the normal component of $V$ to the boundary of $R$ is the volume integral
\[\int_R < \nabla, V > \]

There are versions of Stokes’ theorem for every dimension $n \geq 1$, and other kinds for subsets of curved spaces. The 1-dimensional version is
\[\int_a^b \frac{dy}{dx} dx = y(a) - y(b) \]

namely the fundamental theorem of integral calculus.

Example 4.4.1. Let $R$ be the ball $B$ of radius $r$ centred on the origin $0 \in \mathbb{R}^3$ and define $V$ to be the outwards radial vector field given by
\[V(x) = x \]

The normal component of $V$ to $\partial B$ has magnitude $r$ and so its surface integral is $4\pi r^3$. The divergence of $V$ is
\[< \nabla, V > = 1 + 1 + 1 = 3 \]

and so the volume integral is
\[\frac{4\pi r^3}{3} = 4\pi r^3 \]

\[\square\]
4.5 Differentiation Following Fluid Motion

If $X$ is some property of a fluid its \textit{derivative following the motion of the fluid} $\frac{DX}{Dt}$ is defined to be

$$V_1 \frac{\partial X}{\partial x_1} + V_2 \frac{\partial X}{\partial x_2} + V_3 \frac{\partial X}{\partial x_3} \equiv <V, \nabla > X + \frac{\partial X}{\partial t}$$

where $V$ is the velocity of the fluid flow. Then $\frac{DX}{Dt}$ is the derivative with respect to time of $X$ measured on a single particle moving with the fluid.

The flow is said to be

- \textit{incompressible} when $\frac{D\rho}{Dt} = 0$. In particular, when $\rho$ is constant, in space and time, the flow is \textit{incompressible and homogeneous}

- \textit{steady} when

$$\frac{\partial V}{\partial t} = 0 \text{ and } \frac{\partial \rho}{\partial t} = 0$$

- \textit{non-viscous} when the force on any small body of fluid is always normal to the body. In such a case the pressure $p(x, t)$ at $x \in U$, and at time $t$, is the same in all directions.

Conservation of mass, together with Stokes' theorem, gives the \textit{continuity equation for fluid flow}

$$\rho <\nabla , V > + \frac{D\rho}{Dt} = 0 \quad (4.7)$$

Suppose that the flow is incompressible. Then the continuity equation reduces to

$$<\nabla , V > = 0$$

Let $F : U \times \mathbb{R} \to \mathbb{R}^3$ be the external force field (per unit mass) acting on the fluid. Suppose that $F$ is \textit{conservative} namely

$$F = -\nabla \psi$$

where $\psi : U \times \mathbb{R} \to \mathbb{R}$.

Suppose next, and in addition, that the flow is \textit{nonviscous} and that $U$ is simply-connected. When $V$ is conservative the flow is said to be \textit{irrotational}. It is shown in [2] Section 12 that if the flow is irrotational at any time $t$ then it remains irrotational. In such a case write

$$V = -\nabla \phi$$
4.5. DIFFERENTIATION FOLLOWING FLUID MOTION

where $\phi : U \times \mathbb{R} \rightarrow \mathbb{R}$.

Since $< \nabla, V > = 0$ we have the 3-dimensional Laplace’s equation for $\phi$ at any time $t$

$$\Delta \phi = 0$$  \hspace{1cm} (4.8)

As before, $\Delta$ is the 3-dimensional Laplacian

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

As well as conservation of mass, there is Euler’s equation of motion involving the pressure $p : U \times \mathbb{R} \rightarrow \mathbb{R}$

The equation reads

$$F - \frac{1}{\rho} \nabla p = \frac{DV}{dt} = <V, \nabla > V + \frac{\partial V}{\partial t}$$ \hspace{1cm} (4.9)

Rewriting the right hand side

$$F - \frac{1}{\rho} \nabla p = \frac{1}{2} \nabla ||V||^2 + (\nabla \times V) \times V + \frac{\partial V}{\partial t}$$

Taking account of the conservative nature of $F$ and $V$ the equations simplify further to

$$-\nabla \psi - \frac{1}{\rho} \nabla p = \frac{1}{2} \nabla ||V||^2 = \nabla \frac{\partial \phi}{\partial t}$$ \hspace{1cm} (4.10)

Suppose that at any given time $t$, $\rho$ is a function of $p$ only. Then integration of (4.10) along a line in $U$ gives Bernoulli’s equation

$$-\frac{\partial \phi}{\partial t} + \psi + \int \frac{dp}{\rho} + \frac{1}{2} ||V||^2 = f(t)$$ \hspace{1cm} (4.11)

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is some function of $t$. This simplifies further for steady flow, as in [2] Section 8.

For viscous flow Euler’s equation is replaced by the Navier-Stokes equation

$$F - \frac{1}{\rho} \nabla p + \nu \left( \frac{1}{3} \nabla < \nabla, V > + \Delta V \right) = \frac{DV}{Dt}$$ \hspace{1cm} (4.12)

where $\nu$ is the coefficient of kinematic viscosity. If the flow is incompressible the PDE becomes

$$F - \frac{1}{\rho} \nabla p + \nu \Delta V = \frac{DV}{Dt}$$ \hspace{1cm} (4.13)

Notice
the Navier-Stokes equation (4.12) is of order 2 in $V$ (it involves second order partial derivatives), whereas Euler’s equation (4.9) has order 1

both (4.12) and (4.9) are nonlinear namely the equations are nonlinear in the unknown $V$ and its partial derivatives

we have no special reason to believe viscous flow will be irrotational, even if $F$ is conservative and $\nabla \times V = 0$ at some particular time.

This is all we have to say about fluid flow.

4.6 Static Charges in Free Space

Electric charges exert forces on each other, according to Coulomb’s Law which resembles the inverse-square law for gravitational attraction. A distribution of charge gives rise to a vector field $E$ called the electric field. The vector $E(x)$ is the force exerted on a unit charge placed at $x \in U$ where $U$ is an open subset of $\mathbb{R}^3$.

We look at this in the simplest case, of free space namely a vacuum in which there are no electric or magnetic fields or charges, other than those which we explicitly introduce.

**Example 4.6.1.** If $E$ is due to a point of unit charge located at $x_0 \in \mathbb{R}^3$ in free space, Coulomb’s Law says

$$E(x) = \frac{1}{4\pi \epsilon_0} \frac{x - x_0}{\|x - x_0\|^3}$$

Here

$$\epsilon_0 \approx 8.8541 \times 10^{-12} \text{ coulomb}^2 / \text{N} \cdot \text{m}^2$$

is the permittivity of free space. Notice

1. we’re using Système Internationale (SI) units, in which distances, mass, and length are measured in metres, kilograms and seconds. Besides $\epsilon_0$, there is another constant $\mu_0$ called the permeability of free space which occurs later in the Biot-Savart law for the magnetic induction field $B$. It turns out, essentially from Maxwell’s equations that

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

where $c \approx 299,792,458 \text{ m/s}$ is the speed of light in free space. The relationship between SI units and other systems is discussed in [3] Appendix A
4.6. STATIC CHARGES IN FREE SPACE

2. $E$ is not defined at $x_0$

3. $E$ is conservative with potential function

$$\phi(x) = \frac{1}{4\pi\varepsilon_0} \frac{1}{\|x-x_0\|}$$

Then $\phi : \mathbb{R}^3 - \{x_0\} \to \mathbb{R}$ is called the electric potential, and

$$E = -\nabla \phi$$

where $\nabla$ is the gradient operator for functions defined on open subsets of $\mathbb{R}^3$.

The electric field of a static (time-independent) charge distribution is got by adding the electric fields of point charges that make up the distribution. For a continuous distribution of charge $\rho : U \to \mathbb{R}$ integration over the 3-dimensional region $U$ replaces addition:

$$E = \frac{1}{4\pi\varepsilon_0} \int_U \rho(x_0) \frac{x-x_0}{\|x-x_0\|^3} dx_0$$

Similar things hold for charge distributions on surfaces and along thin wires.

**Example 4.6.2.** Let $W$ be an infinitely long wire with uniform charge density $\rho$ coulombs/m, running along

$$\{(t, 0, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}$$

By symmetry, the electric field $E$ at a point $x \in \mathbb{R}^3$ depends only on the distance $r$ from $x$ to the wire. Furthermore, $E$ points in the unit direction $v$ orthogonal to the wire and in the plane containing $W$ and $x$.

The contribution in the direction of $v$ to $E(x)$, of a single small segment of the wire, starting at $(t, 0, 0)$ and of length $\delta t$, is

$$\frac{\rho}{4\pi\varepsilon_0} \frac{r}{(r^2 + t^2)^{3/2}} \delta t$$

Summing, and taking limits, $E(x)$ is

$$\frac{\rho}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{r}{(r^2 + t^2)^{3/2}} dt = \frac{\rho}{4\pi\varepsilon_0} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = \frac{\rho}{2\pi\varepsilon_0}$$

where $t = r \tan \theta$. □
The electric potential of a charge distribution is obtained in the same way, except then we add scalar valued potential functions rather than vector fields.

\[ \phi = \frac{1}{4\pi\epsilon_0} \int_{U} \frac{\rho(x_0)}{\|x - x_0\|} \, dx_0 \tag{4.14} \]

where integration is over the charge distribution.

Conventional wisdom says this ought to be easier than working with the 3-dimensional vector field \( \mathbf{E} \), but you need to be careful. For instance, a charge distribution has more than one possible potential function. They all differ by constants, and this is sometimes a problem when you come to add the potentials up.

**Exercise 4.6.1.** Use (4.14) to find the electric potential in Example 4.6.2.

Check that \( E(x) = -\nabla \phi \)

Is your answer puzzling? If so, can you identify the problem? \( \square \)

If you had no trouble with Exercise 4.6.1 congratulations: you have sharp eyes! For those who are a little less observant, here’s the source of the difficulty and how to fix it:

**Example 4.6.3.** For a small segment of wire at \((t, 0, 0)\) the contribution to the electric potential is

\[ \frac{\rho}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2 + t^2}} \delta t + b(t) \]

where \(b(t)\) may depend on \(t\) but is at least independent of \(x\). Because we are planning to integrate over a wire \(W\) whose length is infinite, you can see that the integral is quite likely going to be infinite, unless the \(b(t)\) are chosen with special care.

The trick is to choose the \(b(t)\) so that the potentials for each segment all vanish at the same *reference point* in \(\mathbb{R}^3\). Then the integrated contributions be 0 at the reference point, and in particular the integral will be finite. Any reference point will do, provided the potential should be finite there. We choose \((1, 0, 0)\), which is safely off the wire.

Then \(b(t)\) is

\[ -\frac{\rho}{4\pi\epsilon_0} \frac{1}{\sqrt{1 + t^2}} \delta t \]

and \(\phi(x)\) becomes

\[ \frac{\rho}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{r^2 + t^2}} - \frac{1}{\sqrt{1 + t^2}} \, dt = \frac{\rho}{4\pi\epsilon_0} \left[ \ln \frac{\sqrt{r^2 + t^2} + t}{\sqrt{1 + t^2}} \right]_{-\infty}^{\infty} = \]
\[ \frac{\rho}{2\pi \varepsilon_0} (\ln \lim_{t \to \infty} \frac{\sqrt{r^2 + t^2 + t}}{\sqrt{1 + t^2 + t}} - \ln r) \]

Using L’Hôpital’s Rule, the limit becomes

\[ \lim_{t \to \infty} \frac{1 + \frac{t}{\sqrt{r^2 + t^2}}}{1 + \frac{t}{\sqrt{1 + t^2}}} = 1 \]

and therefore

\[ \phi(x) = -\frac{\rho}{2\pi \varepsilon_0} \ln r \]

Now

\[ -\nabla \phi = \frac{\rho}{2\pi \varepsilon_0} \nabla \ln r = \frac{\rho}{2\pi \varepsilon_0 r} \mathbf{v} \]

where \( \mathbf{v} \) is the unit vector in the plane of \( W \) and \( x \), orthogonal to \( W \) and towards \( x \). This agrees with the answer in Example 4.6.2. However we’ve actually had to work harder here with the potential function than directly with the field \( E \) in Example 4.6.2. \( \square \)

So it’s essential (especially when dealing with distributions over infinite regions) for the potentials in (4.14) to all be 0 at a single common reference point in \( \mathbb{R}^3 \).

### 4.7 Electrostatics

Let’s revisit Example 4.6.1.

**Example 4.7.1.** The electric field at \( x \) of a unit charge located at \( x_0 \in \mathbb{R}^3 \) in free space is given by

\[ E(x) = \frac{1}{4\pi \varepsilon_0} \frac{x - x_0}{\|x - x_0\|^3} \]

Then for \( x \neq x_0 \) the divergence \( \langle \nabla, E \rangle (x) \) is the scalar

\[ \frac{1}{4\pi \varepsilon_0} \left( \frac{3}{\|x - x_0\|^3} - \frac{3}{2} \frac{2 < x - x_0, x - x_0 >}{\|x - x_0\|^5} \right) = 0 \]

This remarkable property of the inverse-square law in \( \mathbb{R}^3 \) has important consequences. Notice, by the way that it’s not obvious what happens when \( x = x_0 \) because \( E(x_0) \) is not even defined at \( x_0 \). \( \square \)

If an electric field is due to a charge distribution rather than a single point charge, then in a region where there is no charge it is possible to repeat the calculation in Example 4.7.1. This gives
Proposition 4.7.1. Let $U$ be a region in free space in which there is no charge. Then, even if there is a nonzero electric field $E$ in $U$ (due to a charge distribution outside $U$),
\[ < \nabla, E > = 0 \text{ everywhere on } U \]

Consider next a small ball $B = B(x_0, r)$ of radius $r$ and centre $x_0$ in a region $U \subseteq \mathbb{R}^3$ where there is a charge density
\[ \rho : U \rightarrow \mathbb{R} \]
whose corresponding electric field is $E$. By Proposition 4.7.1, the electric field due to charge outside $B$ contributes nothing to $< \nabla, E > (x)$ when $x \in B$. So the integral of $< \nabla, E >$ over $B$ or, equivalently, the surface integral
\[ S \equiv \int_{\partial B} \frac{< E, x - x_0 >}{\|x - x_0\|} \]
can be calculated as if there is no charge outside $B$.

Exercise 4.7.1. If the only charge within $B$ is a unit point charge at $x_0$ show that $S = \frac{1}{e_0}$.

Working: the integrand is $\frac{1}{r^2}$

Exercise 4.7.2. If the only charge within $B$ is a unit point charge, show that $S = \frac{1}{e_0}$.

Hint: let $x'$ be the location of the charge. Then let $B'$ be the region obtained from $B$ by cutting out a small open ball centred on $x'$. Stokes’ Theorem to $B'$ and recall Exercise 4.7.1.

It follows from Exercise 4.7.2 that for calculating $\int_B < \nabla, E >$, it is not the distribution of charge within $B$ that matters, so much as the total amount of charge. Furthermore, passing to limits, when the charge distribution is given by a charge density $\rho$,
\[ \int_B < \nabla, E > = \frac{1}{\varepsilon_0} \int_B \rho \]
As $r \rightarrow 0+$ this gives$^4$

$^4$For another way of getting this see [3] Section 1.1.3.
Proposition 4.7.2. Gauss’ law In free space an electric field $E$ determined by a static charge distribution $\rho : U \to \mathbb{R}$ satisfies

$$< \nabla, E > = \frac{\rho}{\varepsilon_0} \quad (4.15)$$

Rewriting Gauss’ law in terms of the potential $\phi$ we obtain a second order PDE for $\phi$, called Poisson’s equation

$$\Delta \phi = -\frac{\rho}{\varepsilon_0} \quad (4.16)$$

where $\Delta$ is the 3-dimensional Laplacian

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

When the charge density is 0, Poisson’s equation reduces to Laplace’s equation

$$\Delta \phi = 0 \quad (4.17)$$

Now let’s look briefly at the modifications needed when instead of free space the medium is a material such as air, copper, or glass. As well as the electric field $E$ there is the displacement field $D$. If $\rho$ is the charge density then (4.15) is replaced by an equation for the displacement field $D$.

The electrostatic force on a unit point charge at $x$ is still $E(x)$ and the question then is how to get $E$ from $D$ and known properties of the material. You find yourself considering properties of dipoles to work out the difference between $E$ and $D$. Roughly speaking, the medium tends to absorb some of the charge that would otherwise be used for the electric field. The way in which the charge is absorbed has to do with the polarisation of the medium. This is discussed in some detail in [3] Chapter 7.

For a linear isotropic medium the relationship between $E$ and $D$ is given by

$$D = \varepsilon E \quad \text{where } \varepsilon = \varepsilon_0(1 + \kappa) \text{ and } \kappa \geq 0 \quad (4.18)$$

and the equation replacing (4.15) is

$$< \nabla, V > = \rho \quad (4.19)$$

The constant $\varepsilon$ is the permittivity of the material, and $\kappa$ is the dielectric constant. In free space $\kappa = 0$ and then (4.19) reduces to (4.15).
All this refers to static distributions of charge and static electric fields. When charges and electric fields vary they induce a magnetic inductance $B$. A changing magnetic inductance $B$ in turn causes $E$ to be nonconservative. Equation (4.18) is known as the first constitutive relation, and there are two more that we need to consider. Let’s look at this interesting state of affairs.

### 4.8 Moving Charges

Moving charges are described by currents. The current density at $x$ is a vector field $J(x)$ in the direction of movement of charge. The length of $J(x)$ is the quantity of charge in coulombs/second per unit infinitesimal area at $x$ and orthogonal to $J(x)$. The second constitutional relation is

$$J = \sigma E$$

(4.20)

where $\sigma \geq 0$ depends on the medium. In this Section let’s assume that $J$ is independent of time. Then $J$ is a vector field on some open subset $U$ of $\mathbb{R}^3$. The second constitutional relation (like the first before and the third to come) is not an absolute law of physics. It might happen that $J$ depends nonlinearly on $E$ or that $\sigma$ depends on temperature. The constitutional relations are empirical statements which give reasonable approximations in many situations.

**Example 4.8.1.** For a uniform current flowing through a wire of unit length and unit cross-sectional area, the drop in potential along the wire is

$$V = IR$$

(4.21)

where $I$ is the current, and $R = \frac{1}{\sigma}$. (4.21) is called Ohm’s law, and $\frac{1}{\sigma}$ is the specific resistance of the material.

A basic law of physics is the conservation of charge which says that an increase in charge within a region $R \subset \mathbb{R}^3$ is balanced by charge flowing across the boundary $\partial R$. For a charge density $\rho$ and current density $J$ this amounts to another continuity equation

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

(4.22)

courtesy of Stokes’ theorem.

---

5The first one was for fluid flow.
A current density $J$ induces a magnetic induction field $B$ which in turn exerts a force
$$\mathbf{v} \times B(x)$$
on a unit point charge at $x$ moving with velocity $\mathbf{v} \in \mathbb{R}^3$. Consequently the induction field exerts a force
$$I(x) \times B(x)$$at a point $x$ on a wire carrying a current $I$. The magnetic induction $B$ can be calculated from the *Biot-Savart law*
$$B(x) = \frac{\mu_0}{4\pi} \int \frac{J(x_0) \times (x - x_0)}{\|x - x_0\|^3} \tag{4.23}$$where integration with respect to $x_0$ is over the whole of the region in which $J(x_0) \neq 0$. The constant $\mu_0$ is the *permeability of free space*. It turns out, as a consequence of Maxwell’s equations, and from Maxwell’s description of light as an electromagnetic disturbance, that
$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$where $c$ is the speed of light in free space. We’ll look into this later.

**Exercise 4.8.1.** Prove that
$$\text{div}(B) = 0$$namely that there are *no magnetic sources*. \hfill \Box

The Biot-Savart law, together with the continuity equation (4.22), implies *Ampère’s law*
$$\nabla \times = \mu_0 J \tag{4.24}$$as a mathematical consequence.

Let $S \subset \mathbb{R}^3$ be some smooth surface whose boundary $\partial S$ is a closed curve. Choose, once and for all, a unit normal vector field $N$ to $S$. For $x \in \partial S$ let $\mathbf{w}(x)$ be the unit vector that points *outwards* from $S$. Let $\mathbf{v}(x)$ be the unit vector tangent to $\partial S$ at $x$ such that
$$\mathbf{v}(x) \times \mathbf{w}(x) = N(x)$$
Proposition 4.8.1. For any smooth vector field $V$ defined over $S$, the line integral
\[ \int_{\partial S} <V, N> \]
is the surface integral
\[ \int_S <\nabla \times V, v> \]

So from Ampère’s law, the integral of $B$ around $\partial S$ is the same as the current flowing across $S$.

Example 4.8.2. Suppose a coil is wound many times and let $S$ be a surface transverse to the coil in $n$ places. If a current $I$ is passed through the coil then the current flowing across $S$ is as much as $nI$. The magnetic induction $B$ measured on $\partial S$ is likely to be large if $n$ is large.

All this was for free space. In matter it’s necessary to replace $B$ by the $\mu_0 H$ in Ampère’s law where $H$ is the magnetic field. Then Ampère’s law reads
\[ \nabla \times H = J \] (4.25)

It remains to say what is the relationship between $H$ and $B$. This depends on the medium.

In a linear isotropic medium the relationship is
\[ B = \mu H \] (4.26)

where again the constant $\mu$ is the permeability of the medium.

- for a paramagnetic (respectively diamagnetic) material, $\mu > 1$ (respectively when $\mu < 1$).
- for a ferromagnetic material $\mu \gg 1$, but in fact ferromagnets are usually not linear
- the magnetic field $H$ arises from moving charges. When dielectric effects are also taken into account, we obtain the magnetic induction $B$ which exerts forces on moving charges
- the force on a unit point charge moving with velocity $v$ is still
\[ E + v \times B \]

So the fields $E$ and $B$ are detectable by experiment, whereas $D$ and $H$ are theoretical constructs.
All this is for when the electric displacement $D$ and the magnetic inductance $B$ are independent of time. When either of these is time-varying there are additional effects that need to be taken into account.

### 4.9 Maxwell’s Equations

Michael Farady discovered that a time-varying magnetic inductance produces a current in a loop whose axis is parallel to the variation. More precisely

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad (4.27)$$

This phenomenon can be used to generate electric potential by moving magnets relative to a coil.

Another fact, discovered by James Clerk Maxwell, is that a time-varying electric displacement acts much like a current density, at least so far as Ampère’s law is concerned. So Ampère’s law is modified again to read

$$\text{curl}(H) = J + \frac{\partial D}{\partial t} \quad (4.28)$$

The laws governing electromagnetism can now be summarised as Maxwell’s Equations, namely the following 4 PDEs relating the fields $E, H, D, B, J$ and the charge density $\rho$.

1. $< \nabla, D > = \rho$ (Gauss’ law) \hspace{1cm} (4.29)
2. $< \nabla, B > = 0$ (no magnetic sources) \hspace{1cm} (4.30)
3. $\nabla \times E = -\frac{\partial B}{\partial t}$ (Faraday’s law) \hspace{1cm} (4.31)
4. $\nabla \times H = J + \frac{\partial D}{\partial t}$ (Ampère’s law) \hspace{1cm} (4.32)

Notice that

- since $\text{div} \circ \text{curl} = 0$, it follows from (4.32) that

$$< \nabla, J > + \frac{\partial < \nabla, D >}{\partial t} = 0$$
Substituting for \( \nabla, D \) from (4.29) we obtain
\[
\mathcal{L}_J = 0
\]
which is the continuity equation (4.22).

- because \( \nabla, B \) is 0 we can write
\[
B = \text{curl}(A)
\]
where \( A \) is a time-dependent vector field called the vector potential. Note however that \( A \) is defined only up to sums with vectors whose curl vanishes. So replacing \( A \) by
\[
A + \nabla \alpha \text{ where } \alpha : U \to \mathbb{R}
\]
does not affect \( B \). A change to \( A \) of this sort is called a gauge transformation.

- if \( B \) is nonconstant (4.31) shows that \( E \) is no longer conservative, namely \( E \neq -\nabla \phi \) for a potential function \( \phi \). However we can write
\[
E + \frac{\partial A}{\partial t} = -\nabla \phi
\]
(4.33)
If we require \( \text{div}(A) = 0 \) then, from the constitutive relation (4.18) and (4.29),
\[
\Delta \phi = -\frac{\rho}{\epsilon}
\]
at least in a linear isotropic medium where \( \epsilon \) is constant. The PDE is Poisson’s equation which we met before as (4.16).

Substituting \( B = \nabla \times A \) into Ampère’s Law, and applying the constitutive relations 4.18, 4.26, gives
\[
\nabla \times (\frac{1}{\mu} \nabla \times A) = J + \epsilon \frac{\partial E}{\partial t}
\]

When \( \mu \) is also constant, Exercise 4.4.3 permits us to write
\[
\nabla < \nabla, A > = -\Delta A = \mu J - \epsilon \mu \frac{\partial \nabla \phi}{\partial t} - \mu \epsilon \frac{\partial^2 A}{\partial t^2}
\]
\[\text{There are other ways in which } A \text{ makes itself felt, for instance in the Bohm-Aharonov experiment in quantum physics.}\]
Since \( < \nabla, A > = 0 \)

\[
\Delta A = -\mu J + \epsilon \mu \frac{\partial \nabla \phi}{\partial t} + \epsilon \mu \frac{\partial^2 A}{\partial t^2}
\]

In electrostatics the partial derivatives with respect to \( t \) are 0 and so we are left with Poisson’s Equation for \( A \)

\[
\Delta A = -\mu J
\]

- There is no need to take \( < \nabla, A > = 0 \). As noted already, if \( A \) is replaced by \( A + \nabla \alpha \), where \( \alpha : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) is smooth, then \( \nabla \times A \) is unaffected. However the electric potential \( \phi \) has to be replaced by

\[
\phi - \frac{\partial \alpha}{\partial t}
\]

in order to satisfy (4.33). The requirement \( < \nabla, A > = 0 \) results in the Coulomb gauge.

- The Lorentz gauge is given by

\[
< \nabla, A >= -\epsilon \mu \frac{\partial \phi}{\partial t}
\]

Applying \( \text{div} \) to both sides of (4.33) and substituting in (4.29) we get the inhomogeneous wave equation

\[
\Delta \phi - \epsilon \mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}
\]

Looking at \( \nabla \times (\nabla \times A) \) we obtain

\[
\Delta A - \epsilon \mu \frac{\partial^2 A}{\partial t^2} = -\mu J
\]

which is another inhomogeneous wave equation.

4.10 The Wave Equation

Suppose now that we’re working in free space. Then \( D = \epsilon_0 E, B = \mu_0 H, J = 0 \) and \( \rho = 0 \). Maxwell’s second and third equations therefore read

\[
\nabla \times E = -\frac{\partial B}{\partial t}
\]

(4.34)
\[ \nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} \]  
(4.35)

Taking \textit{curl} of (4.34) and \( \frac{\partial}{\partial t} \) of (4.35)

\[ \nabla \times (\nabla \times E) = -\varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} \]  
(4.36)

From Exercise 4.4.3

\[ \nabla \times (\nabla \times E) = \nabla \cdot \nabla \cdot \varepsilon \nabla \cdot E < E, \nabla > - \Delta E \]  
(4.37)

where on the right the Laplacian \( \Delta \) is applied to each component of the vector field \( E \). So from (4.36) we obtain the (homogeneous) \textit{wave equation}

\[ \Delta E = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} \]  
(4.38)

\textbf{Exercise 4.10.1.} Verify that (4.38) also holds when \( E \) is replaced by \( B \), again in free space.

Notice

- (4.38) says that each of the 3 component functions\(^7\) of \( E \) (and, by Exercise 4.10.1, each of the component functions of \( B \) as well) satisfies the \textit{scalar wave equation}

\[ \Delta y = \varepsilon_0 \mu_0 \frac{\partial^2 y}{\partial t^2} \]  
(4.39)

for a scalar-valued function \( y : U \times \mathbb{R} \rightarrow \mathbb{R} \)

- the wave equation is \textit{linear}, namely linear combinations of solutions with scalar coefficients are also solutions of (4.39). This fact, called the \textit{principle of superposition} is basic to the solution of linear PDEs.

Finding the general solution of the wave equation is more than we can achieve. Obtaining \textit{some solutions}, especially those possessing symmetry in the space variable \( x \), is a task well within our reach.

\textbf{Example 4.10.1.} Let \( \mathbf{v} \) be a nonzero vector in \( \mathbb{R}^3 \) and let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function. Define

\( y : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R} \)

\(^7\)Of course we only say this when \( E \) and \( B \) are expressed using Cartesian coordinates for \( \mathbb{R}^3 \).
by

\[ y(x,t) = f(<x,v> - t) \]

Then for any fixed value of \( t \), \( y(x,t) \) has the same value for all \( x \) lying in any plane normal to \( v \). For \( i = 1, 2, 3 \)

\[ \frac{\partial^2 y}{\partial x_i^2} = f^{(2)}(<x,v> - t)v_i^2 \]

so that

\[ \Delta y = f^{(2)}(<x,v> - t)\|v\|^2 \]

On the other hand

\[ \frac{\partial^2 y}{\partial t^2} = f^{(2)}(<x,v> - t) \]

so that

\[ \Delta y = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \]

where \( c\|v\| = 1 \). Taking \( \|v\|^2 = \epsilon_0\mu_0 \), we see that \( y(x,t) \) satisfies the scalar wave equation (4.39). If \( f \) is replaced by an arbitrary smooth function taking values in \( \mathbb{R}^3 \) then we obtain a solution \( E(x,t) \) of the wave equation (4.38).

Using the same vector \( v \), another smooth vector-valued function \( g : \mathbb{R} \to \mathbb{R}^3 \) determines a solution

\[ B(x,t) = g(<x,v> - t) \]

of the wave equation. Now let’s look at the other conditions that must be satisfied for \( E \) and \( B \) to satisfy Maxwell’s equations in free space. Because \( < \nabla, E >= < \nabla, B >= 0 \), both derivatives \( f^{(1)} \) and \( g^{(1)} \) are everywhere orthogonal to the vector \( v \). Finally we have to satisfy

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

namely

\[ \mathbf{v} \times f^{(1)} = g^{(1)} \]

Since this has to hold everywhere, we can write

\[ g = \mathbf{v} \times f + \text{constant} \]

Summing up

- \( f \) should map into a plane orthogonal to \( v \)
• $g$ should be of the form $\mathbf{v} \times f + \mathbf{w}$ where $\mathbf{w}$ is also orthogonal to $\mathbf{v}$

For instance we might have

$$E(x, t) = E_0 \cos(\omega(\langle x, \mathbf{v} \rangle - t)) + E_1 \sin(\omega(\langle x, \mathbf{v} \rangle - t))$$

and

$$B(x, t) = \mathbf{v} \times E(x, t)$$

where $E_0, E_1 \in \mathbb{R}^3$ are constant vectors orthogonal to $\mathbf{v}$, and where $\omega > 0$. Then $E$ and $B$ are said to define a plane wave whose wavelength is

$$\lambda = \frac{2\pi c}{\omega}$$

Alternatively, the plane wave can be described by taking $E_{\text{const}} \in \mathbb{C}^3$ to be a constant complex-valued vector orthogonal to $\mathbf{v}$ so that the product

$$E(x, t) = E_{\text{const}} e^{i\omega(\langle x, \mathbf{v} \rangle - t)} \in \mathbb{R}^3$$

Plane waves are said to be

• *parallel* when the corresponding vectors $\mathbf{v}$ are parallel,

• *polarised* when they are parallel and the corresponding vectors $E_{\text{const}}$ are parallel in $\mathbb{C}^3$, and

• *in phase* when they are polarised and the corresponding vectors $E_{\text{const}}$ differ by a real scalar factor.

In any case, the electric and magnetic fields remain the same when $(x, 0)$ is replaced by $(x + ct\hat{\mathbf{v}}, t)$ where $\hat{\mathbf{v}}$ is the unit vector in the direction of $\mathbf{v}$. So an electromagnetic disturbance is propagated with speed $c$ in the direction of $\hat{\mathbf{v}}$.

James Clark Maxwell suggested that visible light in free space is an electromagnetic disturbance of this kind. It should follow that

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

where $c$ is the speed of light in free space. This is confirmed experimentally. The *intensity* of the light, in the case of a plane wave, is proportional the square of the length of the vector $E_{\text{const}} \in \mathbb{C}^3$.

Finding the *appropriate* solution of the wave equation, and then Maxwell’s equations, can be difficult. What’s appropriate usually depends on *symmetry* and on *boundary conditions*. For instance, the plane wave solutions of Example 4.10.1 are inappropriate when the electric field $E$ is spherically symmetric about some point in $\mathbb{R}^3$. Rather than go into this right now, let’s look at what happens when current flows through a conducting medium.
4.11 The Telegraphist’s Equation

Suppose still that the charge density $\rho$ is 0 but that current flows through a uniform linear medium of permittivity $\epsilon$ and permeability $\mu$. Then Maxwell’s equations read

\[\langle \nabla, D \rangle = 0 \quad (4.40)\]

\[\langle \nabla, B \rangle = 0 \quad (4.41)\]

\[\nabla \times E = -\frac{\partial B}{\partial t} \quad (4.42)\]

\[\nabla \times H = J + \frac{\partial D}{\partial t} \quad (4.43)\]

Taking curl of (4.42) and $\frac{\partial}{\partial t}$ of (4.43), as in Subsection 4.10, we obtain

\[\nabla \times (\nabla \times E) = -\mu \frac{\partial J}{\partial t} - \epsilon \mu \frac{\partial^2 E}{\partial t^2} \quad (4.44)\]

The constitutive relation (4.20) says $J = \sigma E$, so that

\[\nabla \times (\nabla \times E) = -\mu \sigma \frac{\partial E}{\partial t} - \epsilon \mu \frac{\partial^2 E}{\partial t^2} \quad (4.45)\]

Again using Exercise 4.4.3, we obtain the telegraphist’s equation

\[\Delta E = \epsilon \mu \frac{\partial^2 E}{\partial t^2} + \sigma \mu \frac{\partial E}{\partial t} \quad (4.46)\]

Notice that if $\sigma$ is large, as in a highly conductive medium, then (4.46) resembles the heat equation (4.2).

There are many other interesting applications where partial derivatives are needed to state physical laws. As we have seen, the physical laws generally take the form of partial differential equations (PDEs), namely equations involving partial derivatives of unknown functions of several variables. Next we see how to solve PDEs in one or two simple cases.
CHAPTER 4. PARTIAL DERIVATIVES AND APPLICATIONS
Chapter 5

Partial Differential Equations

Partial differential equations (PDEs) are equations involving partial derivatives of unknown functions of several variables. Ordinary differential equations (ODEs), on the other hand, are equations involving derivatives of unknown functions of a single variable.

Example 5.0.1. The 1-dimensional heat-equation

\[ \frac{\partial y}{\partial t} = c \frac{\partial^2 y}{\partial t^2} \]

is a PDE. Here \( y : [0, L] \times \mathbb{R} \to \mathbb{R} \), and \( y(x, t) \) might be the heat per unit length\(^1\) of a uniform rod at location \((x, 0)\) at time \(t\). The PDE is of second-order in the partial derivatives of \( y \).

The harmonic equation

\[ \frac{d^2 \theta}{dt^2} + \omega^2 \theta \]

is an ODE for \( \theta : \mathbb{R} \to \mathbb{R} \). Here \( \theta(t) \) might be the angle of inclination to the vertical of a simple pendulum\(^2\), and the constant \( \omega^2 \) is given by

\[ \omega^2 = \frac{L}{g} \]

where \( L \) is the length of the pendulum and \( g \) the gravitational acceleration. The ODE is of second-order in the derivatives of \( \theta \).

The following contrasts between PDEs and ODEs are worth noting.

\(^1\)The same PDE holds for the temperature at location \((x, 0)\) and time \(t\).
\(^2\)Actually this is a simplified version of the ODE. The approximation is good when \( \theta \) is small and air-resistance is negligible.
• Typically PDEs have more solutions than ODEs, but they are generally harder to solve.

• The general solution of an $n$th order PDE has $n$ arbitrary functions, whereas for an ODE of the same order there are $n$ arbitrary scalar-valued constants.

• Whereas initial conditions (and sometimes boundary conditions) are needed to find a single solution to an ODE, PDEs almost always require boundary conditions. Boundary-value problems are more difficult to solve than initial-value problems.

So PDEs are worse than ODEs, which seems like very bad news, because ODEs are not easy to solve either. In fact if you write down an ODE at random, you will probably find that Mathematica can’t solve it in terms of known functions.

**Example 5.0.2.** Let’s try

$$\frac{dz}{dt} = z \sin(z^2)$$

The Mathematica command is

```mathematica
DSolve[z'[t] == z*Sin[z[t]^2], z[t], t]
```

Mathematica’s response is

**The equations appear to involve transcendental functions of the variables in an essentially non-algebraic way.**

In fact there is nothing very surprising about this, nor even very discouraging.

• most ODEs can be solved numerically, and

• many ODEs of importance have been well-studied and a lot can be said about their solutions

The most likely problem with the ODE I chose is that it’s of no importance. In effect our language for describing natural phenomena permits us to write about things of no importance. Writing down an ODE without thinking amounts to talking about something that doesn’t exist. When we fail to solve it we get what we deserve, and the same would apply with even greater force if we wrote down a PDE at random.

---

3 which was chosen at random
4 although you never know
5 and is inconvenient for describing some things that are important
Quite a lot is known about ODEs, however, and since PDEs are more complicated a sensible strategy is to

- focus on PDEs which are of significance for applications, in the hope that this will give a better class of PDE, and
- wherever possible, try to reduce PDEs to ODEs.

This doesn’t always work, but it gets us quite a long way. There is of course no shortage of PDEs that are useful for applications. Some examples are given in Chapter 4 and it would not be hard to add to the list. PDEs are probably of even more direct use in applications than ODEs, and PDEs arise naturally in

- Heat Flow,
- Electrostatics,
- Electromagnetism,
- Optics,
- Computer Vision (2D to 3D),
- Geophysics,
- Chemistry,
- Thermodynamics,
- Continuum Mechanics, and
- Fluid Mechanics.

So now we focus on a single simple example from Chapter 4 and try to work out how to solve the PDE.

### 5.1 The Heat Equation

The 1-dimensional heat equation from Section 4.1 has the form

\[
\frac{\partial y}{\partial t} = c \frac{\partial^2 y}{\partial x^2}
\]  

(5.1)
where $c$ is a constant. A partial differential equation such as this has many solutions, and in practice the main difficulty is to find solutions satisfying additional conditions, called boundary conditions. In the case of (5.1) the boundary conditions might be

$$y(0, t) = 0 = y(L, t) \text{ for all } t \geq 0$$

(5.2)

In the situation of Section 4.1 (5.2) says that the heat-density is 0 at the ends of the rod $R$. This would be achieved by the environment acting on the rod.

Another essential piece of information needed to predict $y(x, t)$ is the knowledge of the initial distribution of heat in the rod, namely

$$y(x, 0) \text{ for all } x \in [0, L]$$

(5.3)

This is an initial condition for the PDE (5.1).

Before satisfying (5.2), (5.3) we have a more pressing need, namely to find some solutions of (5.1). At the moment we don’t have any solutions at all, apart from the trivial solution

$$y(x, t) = 0 \text{ for all } x, t$$

This solution is easy enough to spot, but it isn’t much use for anything. So we wait until we have found nontrivial solutions for (5.1) before worrying about the boundary and initial conditions.

The classical approach to generating solutions of a PDE such as (5.1) is to look for solutions of the form

$$y(x, t) = X(x)T(t)$$

(5.4)

where $X$ and $T$ are functions of the single variables $x$ and $t$ respectively. Without loss of generality suppose $T(0) = 1$.

The idea behind this approach is to rewrite (5.1) in terms of functions of single variables. When this is done it is likely that the PDE will reduce to ODEs, and ODEs are likely to be easier to solve. This is exactly what happens, as follows.

A short calculation based on (5.4) gives

$$\frac{\partial y}{\partial t} = X(x)\frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = \frac{d^2 X}{dx^2}T(t)$$

Substituting in (5.1) we get

$$X(x)\frac{dT}{dt} = c\frac{d^2 X}{dx}T(t)$$
5.1. THE HEAT EQUATION

\[ \frac{1}{T(t)} \frac{dT}{dt} = \frac{c}{X(x)} \frac{d^2X}{dx^2} \quad (5.5) \]

assuming neither \( T(t) \) nor \( X(x) \) is 0.

Notice that (5.5) has achieved something very like separation of variables for ODEs, namely the left hand side is independent of \( x \) and the right hand side is independent of \( t \). Since the left and right hand sides agree they must both be independent of \( x \) and \( t \), namely they are both constant. Let \( \gamma \in \mathbb{R} \) be the value of this constant.

Then (5.5) can be rewritten as the pair of ODEs

\[ \frac{1}{T(t)} \frac{dT}{dt} = \gamma \quad (5.6) \]

and

\[ \frac{c}{X(x)} \frac{d^2X}{dx^2} = \gamma \quad (5.7) \]

To solve (5.6) integrate both sides with respect to \( t \). Then

\[ \ln T = \gamma t \]

where \( T(t) > 0 \), and the constant of integration is 0, because \( T(0) = 1 \). Then

\[ T(t) = e^{\gamma t} \quad (5.8) \]

If \( \gamma > 0 \) then

\[ \lim_{t \to \infty} T(t) = \infty \]

and if \( X(x) \neq 0 \) for some \( x \) then

\[ \lim_{t \to \infty} |y(x,t)| = \infty \]

This is inconsistent with our intuitive notions about the flow of heat and we assume from now on that \( \gamma \leq 0 \). Write

\[ \gamma = -\alpha^2 c \]

Then

\[ T(t) = e^{-\alpha^2 ct} \quad (5.9) \]
and (5.7) becomes
\[ \frac{d^2 X}{dx^2} + \alpha^2 X = 0 \]
whose general solution is
\[ X(x) = a \cos(\alpha x) + b \sin(\alpha x) \quad (5.10) \]
where \( a, b \in \mathbb{R} \) are arbitrary constants. Substituting in (5.4) we obtain
\[ y(x, t) = (a \cos(\alpha x) + b \sin(\alpha x)) e^{-\alpha^2 c t} \quad (5.11) \]
Since (5.11) is a solution we can try to satisfy the boundary conditions (5.2). From these we obtain
\[ a = 0 \text{ and } b \sin(\alpha L) = 0 \]
To obtain a nontrivial solution satisfying (5.2) we need to set
\[ \alpha L = n \pi \text{ where } n \in \mathbb{Z}_+ \]
Then
\[ y(x, t) = b \sin\left(\frac{n \pi x}{L}\right) e^{-\alpha^2 c t} \quad (5.12) \]
In all likelihood this solution will not satisfy the initial condition (5.3)
\[ y(x, 0) = g(x) \text{ for all } x \]
At this point we can make two useful observations.

- (5.1) is a linear PDE, namely if \( y_1, y_2 : [0, L] \times \mathbb{R}_+ \rightarrow \mathbb{R} \) both satisfy (5.1) then, for any \( b_1, b_2 \in \mathbb{R} \), the linear combination
  \[ b_1 y_1 + b_2 y_2 \]
is also a solution of (5.1).

Exercise 5.1.1. Verify this assertion. \( \square \)

It follows similarly that if
\[ y_1, y_2, \ldots y_n, \ldots \]
is any sequence of solutions of (5.1), and \( b_1, b_2, \ldots b_n, \ldots \) any sequence of coefficients, then provided the series converges in a suitable sense
\[ y = \sum_{n \geq 1} b_n y_n \]
is also a solution of (5.1).
5.2. THE WAVE EQUATION

- The boundary conditions (5.2) are \textit{homogeneous} namely that if \( y_n \) satisfies (5.2) for every \( n \geq 1 \) then \( y \) also satisfies (5.2).

For \( n \geq 1 \) set

\[
y_n(x, t) = \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 ct}{L^2}}
\]

so that

\[
y(x, t) = \sum_{n \geq 1} b_n \sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 ct}{L^2}} \tag{5.13}
\]

Our problem now is to choose the coefficients \( b_1, b_2, \ldots, b_n, \ldots \) so that the initial condition (5.3) is satisfied, namely

\[
\sum_{n \geq 1} b_n \sin\left(\frac{n\pi x}{L}\right) = g(x) \quad \text{for all } x \in [0, L] \tag{5.14}
\]

Next make a change of variables \( \bar{x} = \frac{x}{L} \), so that (5.30) becomes

\[
\tilde{g}(\bar{x}) \equiv g\left(\frac{L\bar{x}}{\pi}\right) = \sum_{n \geq 1} b_n \sin(n\bar{x}) = \quad \text{for all } \bar{x} \in [0, \pi] \tag{5.15}
\]

Extend \( \tilde{g} \) to an odd function defined over the whole of \([-\pi, \pi]\). Taking \( b_1, b_2, \ldots, b_n, \ldots \) to be the Fourier coefficients

\[
b_n = \frac{2}{\pi} \int_0^\pi \tilde{g}(\bar{x})d\bar{x}
\]

of \( \tilde{g} \), we see that (5.15) is satisfied, at least when \( \tilde{g} \) is continuous. Substituting these values for the coefficients \( b_n \) into (5.13) we obtain a solution \( y \) of the heat equation (5.1) satisfying the boundary conditions (5.2) and the initial condition (5.3).

This was what Fourier had in mind when he invented Fourier series, as a way of solving the heat equation. It counts as a substantial achievement on its own, but there is more good news. There are other important examples of PDEs where similar methods apply.

5.2 The Wave equation

The \textit{1-dimensional wave equation} from Section 4.2 has the form

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{5.16}
\]
where $c$ is a constant. This looks similar to the heat equation (5.1) but the solutions of the wave equation turn out to be quite different.

As described in Section 4.2, the wave equation models the height $y(x, t)$ at time $t$ above $(x, 0)$, of an elastic string stretched between $(0, 0)$ and $(0, L)$. The boundary conditions in such a case are

$$y(0, t) = 0 = y(L, t) \text{ for all } t \geq 0 \quad (5.17)$$

which happen to have the same form as for the heat equation. An initial condition is

for some given function $g : [0, L] \rightarrow \mathbb{R}$, $y(x, 0)$ for all $x \in [0, L] \quad (5.18)$

which means that the shape of the string is known at $t = 0$. This initial condition has the same form as (5.3), but in the case of the wave equation it is necessary\(^6\) to know an additional piece of information.

Imagine that the string is plucked at time $t = 0$. It is not only the initial shape of the string that is important if we are to predict $y(x, t)$. Another thing that matters is the way in which the string is moving at $t = 0$. So another initial condition is needed, and we suppose

$$\frac{\partial y}{\partial t} \big|_{(x, 0)} = 0 \text{ for all } x \in [0, L] \quad (5.19)$$

As before, we look for solutions of the form

$$y(x, t) = X(x)T(t) \quad (5.20)$$

where $X$ and $T$ are functions of the single variables $x$ and $t$ respectively.

Without loss $T(0) = 1$.

Then

$$\frac{\partial^2 y}{\partial t^2} = X(x) \frac{d^2 T}{dt^2} \text{ and } \frac{\partial^2 y}{\partial x^2} = \frac{d^2 X}{dx} T(t)$$

Substituting in (5.16)

$$X(x) \frac{d^2 T}{dt^2} = c^2 \frac{d^2 X}{dx} T(t)$$

namely

$$\frac{1}{T(t)} \frac{d^2 T}{dt^2} = \frac{c^2}{X(x)} \frac{d^2 X}{dx^2} \quad (5.21)$$

\(^6\) This is traceable to the fact that (5.16) is of second order in the partial derivatives with respect to $t$.\)
5.2. THE WAVE EQUATION

assuming neither $T(t)$ nor $X(x)$ is 0.

The left hand side of (5.21) is independent of $x$ and the right hand side is independent of $t$. Since the left and right hand sides agree they must both be independent of $x$ and $t$, namely they are both constant. Let $\gamma \in \mathbb{R}$ be the constant.

Then (5.21) can be rewritten as the pair of ODEs

$$\frac{1}{T(t)} \frac{d^2 T}{dt^2} = \gamma \quad (5.22)$$

and

$$\frac{c^2}{X(x)} \frac{d^2 X}{dx^2} = \gamma \quad (5.23)$$

**Exercise 5.2.1.** Show that if $\gamma > 0$ then any nontrivial solution $T$ of (5.22) satisfies

$$\lim_{t \to \infty} |T(t)| = \infty$$

Conclude that $\gamma = -\alpha^2$ for some real constant $\alpha$, at least in the case of a vibrating string.

Writing $\gamma = -\alpha^2 c^2$, the general solutions to the second order ODEs are

$$T(t) = d \cos(\alpha t) + e \sin(\alpha t) \quad (5.24)$$

and

$$X(x) = a \cos(\alpha x) + b \sin(\alpha x) \quad (5.25)$$

where $a, b, d, e \in \mathbb{R}$ are arbitrary constants.

However $T(0) = 1$, namely $d = 1$ and

$$\frac{\partial y}{\partial t}|_{(x,0)} = X(x) \frac{dT}{dt}|_{t=0}$$

Assuming $X(x) \neq 0$ for some $x \in [0, L]$ it follows from the second initial condition (5.19) that

$$\frac{dT}{dt}|_{t=0} = 0$$

namely $e = 0$. So we can write

$$T(t) = \cos(\alpha t) \quad (5.26)$$
Substituting in (5.4w) we obtain
\[ y(x, t) = (a \cos(\alpha x) + b \sin(\alpha x)) \cos(\alpha c t) \] (5.27)
To satisfy the boundary conditions (5.17)
\[ a = 0 \text{ and } b \sin(\alpha L) = 0 \]
so that for a nontrivial solution
\[ \alpha L = n\pi \text{ where } n \geq 0 \text{ is an integer} \]
Then
\[ y(x, t) = b \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right) \] (5.28)
We still haven’t satisfied the initial condition (5.18)
\[ y(x, 0) = g(x) \text{ for all } x \]
Note now that

- (5.16) is another linear PDE, namely if
  \[ y_1, y_2, \ldots y_n, \ldots \]
is a sequence of solutions of (5.16), and \( b_1, b_2, \ldots b_n, \ldots \) a sequence of coefficients, then provided the series converges in a suitable sense
\[ y = \sum_{n \geq 1} b_n y_n \]
is also a solution of (??).

- The boundary conditions (5.17) are homogeneous\(^7\) namely that if \( y_n \) satisfies (5.17) for every \( n \geq 1 \) then \( y \) also satisfies (5.17).

For \( n \geq 1 \) set
\[ y_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right) \]
so that
\[ y(x, t) = \sum_{n \geq 1} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right) \] (5.29)

\(^7\)In fact these are the same as the boundary conditions (5.2) for the heat equation.
As for the heat equation, we now need to choose $b_1, b_2, \ldots, b_n, \ldots$ so that (5.18) is satisfied, namely

$$\Sigma_{n \geq 1} b_n \sin\left(\frac{n\pi x}{L}\right) = g(x) \text{ for all } x \in [0, L] \quad (5.30)$$

This is done in exactly the same way as for the heat equation, namely set $\bar{x} = \frac{x}{L}$, $\bar{g}(\bar{x}) = g\left(L\bar{x}\right)$ and

$$b_n = \frac{2}{\pi} \int_0^\pi \bar{g}(\bar{x}) d\bar{x}$$

If you would like to see how this works in a particular case, go to

http://maths.uwa.edu.au/~lyle/cm/2CA1/string.html
Appendix A

Different Conventions

A.1 Fourier Sums and Series

Our Fourier Sums and Fourier Series are for real or complex valued functions $y$ defined on an interval $[-\pi, \pi)$. The Fourier Series then has the form

$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx$$

The Fourier Exponential Series is

$$ES(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) e^{-inx} dx$$

When Fourier Series are studied for functions on different intervals it’s necessary to change these formula. In general, for $y$ defined on $[a, b)$ where $a < b$

$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi x}{b-a}) + b_n \sin(\frac{2n\pi x}{b-a})$$
where

\[ a_n = \frac{2}{b-a} \int_a^b y(x) \cos\left(\frac{2\pi n x}{b-a}\right) dx \quad \text{and} \quad b_n = \frac{2}{b-a} \int_a^b y(x) \sin\left(\frac{2\pi n x}{b-a}\right) dx \]

Similarly, the Fourier Exponential Series for a function \( y \) defined on \([a, b)\) is

\[ ES(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i x}{b-a}} \]

where

\[ c_n = \frac{1}{b-a} \int_a^b y(x) e^{-\frac{2\pi i x}{b-a}} dx \]
Bibliography


Index

nth partial sum, 66
1-dimensional heat equation, 115
Banach space, 72
Cauchy, 62
Dirichlet-Jordan Theorem, 77
Euclidean distance, 44
Euclidean length, 44
Euclidean norm, 61
Euler’s equation of motion, 95
Fourier series, 64
Fourier sum, 9
Gibbs phenomenon, 79
Gram-Schmidt Orthogonalization, 48
Heaviside function, 69, 78
Hermitian norm, 62
Hilbert space, 73, 80
Kronecker delta, 13
Laplacian, 88
Ordinary differential equations, 113
Parseval’s Identity, 6
Parseval’s identity, 79
Partial differential equations, 113
Poisson’s equation, 106
Riemann integral, 69
Riemann sums, 69
Riemann-Lebesgue Lemma, 73
Taylor’s theorem, 56
alternating, 60
arithmetic series, 53
associated norm, 46
boundary conditions, 114
bounded variation, 77
common difference, 54
common ratio, 54
complete, 62
completion, 69
conservative, 90
converge, 54, 58
convergence, 52
convergent, 59
converges, 55
convolution, 81, 85
convolution equations, 84
curl, 91
data compression, 7
data smoothing, 7
divergent, 59
doubly-infinite sequence, 65
doubly-infinite sequences, 63
even function, 18
exponential series, 64
exponential sum, 10
function spaces, 6
geometric series, 54
gradient, 89
harmonic equation, 113
harmonic series, 55, 60
heat equation, 89
image compression, 7
incomplete, 62
infinite series, 6, 60
inner product, 6, 45
irrotational, 91
isometry, 81
lemma, 13
measure zero, 72, 80
INDEX

nonlinear, 96
norm, 6, 43
normed vector spaces, 6
odd function, 19
of measure zero, 69
partial differential equations, 5, 85, 87, 111
partial sum, 60, 74
periodic of period, 9
pointwise convergence, 64
potential function, 90
power series, 55, 56, 64
projection, 36
propositions, 13
second-order, 113
sequence, 58
series, 60
specific heat, 89
square-integrable, 73, 79
the trivial vector subspace, 30
theorems, 13
triangle inequality, 44, 46
trigonometric series, 55
trigonometric series, 57, 64
trigonometric sums, 6
trivial vector space, 28
uniform convergence, 65
uniform norm, 44, 65
wave equation, 87, 88
2-dimensional Laplacian, 116
3-dimensional Laplacian, 95
Ampère’s law, 103
Bernoulli’s equation, 95
Biot-Savart law, 96, 103
Bohm-Aharonov experiment, 106
Cauchy Data, 116, 117
Cauchy Problem, 117
Cauchy-Riemann Equations, 116
Coulomb gauge, 107
Coulomb’s Law, 96
L’Hôpital’s Rule, 99
Laplace’s Equation for a real-valued function on \( \mathbb{R}^2 \), 116
Laplace’s equation, 94, 101
Laplacian, 101
Lorentz gauge, 107
Maxwell’s Equations, 105
Maxwell’s equations, 96
Method of Characteristic Strips, 117
Navier-Stokes equation, 95
Ohm’s law, 102
Poisson’s equation, 101
Stokes’ theorem, 93
Système Internationale, 96
characteristic curves, 119
characteristic strips, 118
characteristic, 120
coefficient of kinematic viscosity, 95
complex analytic function, 116
conservation of charge, 102
conservative, 94, 97
costitutional relation, 102
continuity equation for fluid flow, 94
continuity equation, 102
continuous distribution of charge, 97
current density, 102
derivative following the motion of the fluid, 93
diamagnetic, 104
dielectric constant, 101
dipoles, 101
displacement field, 101
divergence, 92
electric field, 96
electric potential, 97
ferromagnetic, 104
first-order partial differential equation, 117
free space, 96
gauge transformation, 106
heat equation, 111
in phase, 110
incompressible and homogeneous, 94
incompressible, 94
irrotational, 94
linear isotropic medium, 101, 104
magnetic field, 104
magnetic inductance, 102
magnetic induction field, 103
magnetic induction, 96
magnetic induction, 104
multi-index, 116
non-viscous, 94
nonviscous, 94
parallel, 110
paramagnetic, 104
partial differential equation of order m, 116
permeability of free space, 96, 103
permeability, 104
permittivity of free space, 96
permittivity, 101
plane wave, 110
polarisation, 101
polarised, 110
pressure, 94
principle of superposition, 108
quasilinear partial differential equation, 119
reference point, 98
specific resistance, 102
steady, 94
telegraphist’s equation, 111
vector potential, 106
viscous flow, 95
wave equation, 108
wavelength, 110
constitutive relation, 102