This mock-exam resembles the actual exam in its content (but not necessarily details). Like the actual exam it has 4 questions of equal value, and is to be sat in 2 hours. Solutions can be found either in lecture notes or from similar examples done in practice sessions or tests. Notes and calculators are not permitted in the actual exam.

Students will be supplied with copies of trigonometric formulae.

Attempt as many questions as you can.

As usual, \( i^2 = -1 \), and \( e \) stands for the number 2.71828... whose natural logarithm is 1.

1. Let \( V \) be the complex vector space \( \ell^2 \) of doubly-infinite sequences

\[
\{ c_n \in \mathbb{C} : n \in \mathbb{Z} \text{ and } \Sigma_{n \in \mathbb{Z}} |c_n|^2 < \infty \}
\]

For \( c, d \in V \) define

\[
\langle c, d \rangle = \Sigma_{n \in \mathbb{Z}} c_n \overline{d_n}.
\]

(a) Prove that \( \langle , \rangle \) is an inner product on \( V \).

(b) For \( m \in \mathbb{Z} \) define \( e^m \in V \) by

\[
e^m_n = \delta^m_n
\]

Prove that

\[
\{ e^m : m \in \mathbb{Z} \}
\]

is an orthonormal subset of \( V \).

(c) Let \( y(x) = x^2 \) for \( x \in [-\pi, \pi] \). Find the Fourier series of \( y \). Conclude that

\[
1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \frac{\pi^2}{12}.
\]
(d) State Parseval's identity, and apply it to the Fourier series of \( y \) in (c).

2. (a) Let \( y, z : \mathbb{R} \to \mathbb{C} \) be piecewise-continuous periodic functions of period 2\( \pi \). The convolution \( y \ast z : \mathbb{R} \to \mathbb{C} \) is defined by

\[
(y \ast z)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x-u)z(u)du
\]

(i) How are the coefficients of the exponential series of \( y, z, y \ast z \) related? Give your reasoning.

(ii) Let \( y(x) = x \) and \( z(x) = 1 \) for \( x \in [-\pi, \pi) \). Calculate \( (y \ast z)(x) \) for \( 0 < x < \pi \).

Caution: remember to treat \( y \) and \( z \) as periodic outside \([-\pi, \pi)\).

(b) Explain how Fourier series can be used in the solution of the partial differential equation

\[
\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}
\]

subject to the boundary conditions

\[
f(0, t) = 0 = f(\pi, t)
\]

and the initial condition

\[
f(x, 0) = y(x) \text{ for all } x \in (0, \pi)
\]

where \( f \) is an unknown function of 2 variables \( x \) and \( t \), and \( y : [0, \pi] \to \mathbb{R} \) is a given continuous function with \( y(0) = 0 = y(\pi) \). (It will be enough to describe the main steps and show how Fourier series are used.)

3. (a) Let \( y(x) = 1 - |x| \) for \( |x| < 1 \), and let \( y(x) = 0 \) otherwise. Show that the Fourier transform of \( y \) is given by

\[
\tilde{y}(\xi) = \frac{2\sin^2(\pi\xi/2)}{\pi^2\xi^2} \text{ for } \xi \neq 0.
\]

(b) Write down the Poisson summation formula.

(c) Prove that

\[
\sum_{m \in \mathbb{Z}} e^{-2m^2\pi^2} = \sqrt{\frac{1}{2\pi}} \sum_{m \in \mathbb{Z}} e^{-\frac{m^2}{\pi}}.
\]
4. (a) Show how Fourier transforms can be viewed as limits of exponential series.

(b) Suppose that \( f : \mathbb{R}^2 \to \mathbb{C} \) satisfies the PDE

\[
\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}
\]

as well as the condition

\[
f(x, 0) = y(x)
\]

where \( y : \mathbb{R} \to \mathbb{C} \) is a given \( L^1 \) function.

(i) Sketch a proof that the Fourier transform \( \tilde{f} \) of \( f \) with respect to the \( x \) variable satisfies

\[
\tilde{f}(\xi, t) = \tilde{y}(\xi)e^{-\pi^2 \xi^2 t}.
\]

(ii) Show that

\[
f(x, t) = \int_{-\infty}^{\infty} e^{\pi i x \xi - \pi^2 \xi^2 t} \tilde{y}(\xi) d\xi.
\]

(Ignore questions of regularity.)