1 Solutions to Exercise Sheet 1

1. For a bounded continuous function $y : [-\pi, \pi) \to \mathbb{C}$ let $a_n, b_n \in \mathbb{C}$ be the Fourier coefficients of $y$. Let $c_n \in \mathbb{C}$ be the coefficients of the corresponding exponential sum. How are the $c_n$ related to the $a_n$ and $b_n$? Show your working.

The Fourier sum of $y$ has the form

$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

and the corresponding exponential sum is

$$ES(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $FS(x) = ES(x)$ for all $x \in [-\pi, \pi)$. From lectures, where we derived a formula for the coefficients $c_n$, the $c_n$ are unique, determined only by $y$, since $y$ is continuous.

Substituting the known formulae (from lectures)

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2} \quad \text{and} \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

into $FS(x)$ we get

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n}{2} - i \frac{b_n}{2} \right) e^{inx} + \sum_{n=1}^{\infty} \left( \frac{a_n}{2} + i \frac{b_n}{2} \right) e^{-inx} =$$

$$\sum_{n=-\infty}^{\infty} \left( \frac{a_n}{2} + i \frac{b_n}{2} \right) e^{inx} + \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n}{2} - i \frac{b_n}{2} \right) e^{inx}$$

This has the same form as $ES(x)$. Since $FS(x) = ES(x)$ and the coefficients of $ES(x)$ are unique, it follows that $c_n$ is

$$\frac{a_n}{2} + i \frac{b_n}{2}, \quad \frac{a_0}{2}, \quad \text{or} \quad \frac{a_n}{2} - i \frac{b_n}{2}$$

according as $n < 0$, $n = 0$, or $n > 0$. We can also use these equations to solve for $a_n, b_n$ in terms of $c_n$, namely

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad \text{and} \quad b_n = i(c_n - c_{-n})$$

where $n > 1$. 

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2. In question 1 above, give necessary and sufficient conditions on the $c_n$ for the $a_n$ and $b_n$ to be real numbers.

If the $a_n$ and $b_n$ are real then, from the expressions for $c_n$ in terms of $a_n, b_n$ in question (1) above,

- $c_0 = \frac{a_0}{2}$ is real, and

- $\bar{c}_n = \frac{a_n}{2} + \frac{i b_n}{2} = c_{-n}$

From the expressions for the $a_n, b_n$ in terms of the $c_n$, in question (1) above we see that

- if $c_0$ is real so is $a_0$, and

- if $\bar{c}_n = c_{-n}$ then

\[
\bar{a}_n = c_{-n} + c_n = a_n \quad \text{and} \quad \bar{b}_n = -i(c_{-n} - c_n) = b_n
\]

and so $a_n$ and $b_n$ are real where $n > 1$.

3. Prove directly, without reference to the $c_n$, that the Fourier coefficients $a_n, b_n$ are

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx
\]

where $n \geq 0$ and $n \geq 1$ respectively.

If $y(x)$ is a function of the form

\[
a_0 + \sum_{p=1}^{m} (a_p \cos(px) + b_p \sin(px))
\]

then the coefficients $a_0$ and $a_n, b_n$ are called the Fourier coefficients\(^2\). In the case of such a function

\[
\int_{-\pi}^{\pi} \cos(nx)dx = \frac{2}{\pi} \sin(nx)
\]

\(^1\)Because $n$ appears in the quoted formulae we use $p$ as a label for the dummy variable instead of $n$.

\(^2\)The formulae quoted above are proved in lectures using similar formulae for the $c_n$ in the case of a trigonometric sum, and then defined for arbitrary functions for which the integral is defined using the quoted formulae.
\[ \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(nx) \, dx + \sum_{p=1}^{m} a_p \int_{-\pi}^{\pi} \cos(px) \cos(nx) \, dx + \sum_{p=1}^{m} b_p \int_{-\pi}^{\pi} \sin(px) \cos(nx) \, dx \]

- if \( n = 0 \) then

\[ \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(nx) \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \, dx = \pi a_0 \]

and for \( p > 0 \)

\[ \int_{-\pi}^{\pi} \cos(px) \cos(nx) \, dx = \int_{-\pi}^{\pi} \cos(px) \, dx = \]

\[ \frac{1}{p} (\sin(p\pi) - \sin(-p\pi)) = \frac{1}{p} (0 - 0) = 0 \]

Similarly

\[ \int_{-\pi}^{\pi} \sin(px) \cos(nx) \, dx = \int_{-\pi}^{\pi} \sin(px) \, dx = \]

\[ -\frac{1}{p} (\cos(p\pi) - \cos(-p\pi)) = -\frac{1}{p} ((-1)^p - (-1)^p) = 0 \]

Therefore

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(0x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \, dx \]

as stated.

- for \( n > 0 \) we have \( \int_{-\pi}^{\pi} \cos(nx) \, dx = 0 \), since this was proved above with \( p > 0 \) in place of \( n \). Now\(^3\)

\[ 2 \cos(px) \cos(nx) = \cos(p-n)x + \cos(p+n)x \]

and therefore

\[ \int_{-\pi}^{\pi} \cos(px) \cos(nx) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(p-n)x + \cos(p+n)x \, dx \]

Since \( p+n \in \mathbb{Z}_+ \) the integral of the second term is 0, and the only way \( \int_{-\pi}^{\pi} \cos(p-n)x \, dx \neq 0 \) is for \( p = n \) in which case the integral is \( 2\pi \).

\(^3\text{You are expected to know such formulae from school.}\)
\( \sin(px) \cos(nx) \) is an odd function and therefore the integral
\[
\int_{-\pi}^{\pi} \sin(px) \cos(nx) \, dx
\]
is 0. Therefore
\[
\int_{-\pi}^{\pi} y(x) \cos(nx) \, dx = \pi a_n
\]
as claimed.

This proves the formulae for \( a_n \). The formula for \( b_n \) is proved similarly, using the facts that
\begin{itemize}
  \item \( \sin(nx) \) and \( \cos(px) \sin(nx) \) are odd functions, and
  \item \( 2 \sin(px) \sin(nx) = \cos(p-n)x - \cos(p+n)x \)
\end{itemize}

4. Let \( y \) be an even function and \( z \) an odd function. Show that \( yz \) is an odd function.

\[
(yz)(-x) = y(-x)z(-x) = -y(x)z(x)
\]
and so \( yz \) is odd.

5. Let \( y \) and \( z \) both be even functions. Show that \( yz \) is also even.

\[
(yz)(-x) = y(-x)z(-x) = y(x)z(x) = y(x)z(x)
\]
and so \( yz \) is even.

6. Let \( y \) and \( z \) both be odd functions. Show that \( yz \) is even.

\[
(yz)(-x) = y(-x)z(-x) = (-1)^2y(x)z(x) = y(x)z(x)
\]
and so \( yz \) is even.

7. If \( y \) is an even function show that
\[
\int_{-\pi}^{\pi} y(x) \, dx = 2 \int_{0}^{\pi} y(x) \, dx
\]
\[
\int_{-\pi}^{\pi} y(x) \, dx = \int_{-\pi}^{0} y(x) \, dx + \int_{0}^{\pi} y(x) \, dx = \]
\[-\int_{-\pi}^{\pi} y(u) du + \int_{0}^{\pi} y(x) dx\]

where we made the substitution \(u = -x\) and used the fact that \(y(x) = y(-x)\). The right hand side can now be rewritten

\[\int_{0}^{\pi} y(u) du + \int_{0}^{\pi} y(x) dx = 2 \int_{-\pi}^{\pi} y(x) dx\]

8. If \(y\) is an odd function show that

\[\int_{-\pi}^{\pi} y(x) dx = 0\]

\[\int_{-\pi}^{\pi} y(x) dx = \int_{-\pi}^{0} y(x) dx + \int_{0}^{\pi} y(x) dx = \int_{\pi}^{0} y(u) du + \int_{0}^{\pi} y(x) dx\]

where we made the substitution \(u = -x\) and used the fact that \(y(x) = -y(-x)\). The right hand side can now be rewritten

\[-\int_{0}^{\pi} y(u) du + \int_{0}^{\pi} y(x) dx = 0\]

Using Exercises 4, 5, 6, 7, 8:

9. Let \(y : [-\pi, \pi] \to \mathbb{R}\) be an odd function. Show that

\[FS(x) = b_1 \sin x + b_2 \sin(2x) + \ldots + b_m \sin(mx)\]

where for \(1 \leq n \leq m\)

\[b_n = \frac{2}{\pi} \int_{0}^{\pi} y(x) \sin(nx) dx\]

Because \(y\) is an odd function of \(x\) so is \(y(x) \cos(px)\) for \(p > 0\), by Exercise 4. Therefore \(a_p = 0\) for all \(p \geq 0\), by Exercise 8. By Exercise 6 \(\sin(px) \sin(nx)\) is even and therefore \(b_n\) can be written as above, using Exercise 7.
10. Experiment with \( y = x^3 \) to derive an identity involving \( \pi \).

The integrals can be worked out directly, but this is an occasion where you ought to use Mathematica. After consulting the Help Browser\(^4\) load the FourierTransforms package thus:

\[ \text{<<Calculus'FourierTransform'} \]

and then

\[ \text{FourierSinSeriesCoefficient}[x^3,\{x,-\pi,\pi\},n] \]

to get

\[ b_n = \frac{2(-1)^n}{n^3} (6 - n^2 \pi^2) \]

Since \( x^3 \) is continuous we have

\[ \frac{x^3}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} (6 - n^2 \pi^2) \sin(nx) \]

Taking \( x = \frac{\pi}{2} \), \( \frac{\pi^3}{16} \) is

\[ -6(1 - \frac{1}{27} + \frac{1}{125} + \ldots) \]

\[ \pi^2(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots) \]

Notice that the right hand side of this expression has been evaluated elsewhere. In Example 1.6.2 of the web-notes

\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4} \]

and in Example 1.6.3

\[ 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{32} \]

Substituting for these expressions we now obtain, as confirmation of our working,

\[ \frac{\pi^3}{16} = -6 \frac{\pi^3}{32} + \frac{\pi^3}{4} \]

\(^4\)Look up FourierSinSeriesCoefficients