1 Solutions to Exercise Sheet 2

1. Check that the conditions for $V$ to be a real or complex vector space imply the additional condition

`for every $v \in V$ there is an element $-v \in V$ with the property that $-v + v = 0`'

Let $w = (-1)v$. Then

$$w + v = (-1)v + (1)v = (-1 + 1)v = 0v = 0$$

So take $-v$ to be $(-1)v$.

For any (real or complex) vector space $V$, a linear combination of $v_1, v_2, \ldots v_m \in V$ is an expression of the form

$$t_1v_1 + t_2v_2 + \ldots t_mv_m \in V$$

where $t_1, t_2, \ldots t_m$ are scalars. The set of all linear combinations of elements of a nonempty subset $S$ of $V$ is called the span $\text{span}(S) = <S>$ of $S$.

2. Prove that the span of any nonempty subset (not necessarily finite) of a vector space $V$ is a vector subspace of $V$.

We need to check first that, with respect to the operations of vector addition and scalar multiplication for $V$

$$v + w \in <S> \text{ and } t.v \in <S>$$

whenever $v, w \in V$ and $t$ is any scalar, namely $<S>$ should be closed with respect to vector addition and scalar multiplication. This is so that $+$ and . restrict to operations on $<S>$ (there is always the chance that they might produce elements of $V$ that are not in $S$. Write $v, w \in <S>$ in the form

$$t_1v_1 + t_2v_2 + \ldots t_mv_m$$

and

$$t_{m+1}v_{m+1} + t_{m+2}v_{m+2} + \ldots t_nv_n$$

respectively, where $t_1, t_2, \ldots t_n$ are scalars and $v_1, v_2, \ldots v_n \in S$. Possibly there are some repetitions among the $v_i$ but this doesn’t matter. Then

$$v + w = \Sigma_{i=1}^{n} t_iv_i \in <S>$$

and

$$t.v = \Sigma_{i=1}^{m} (tt_i)v_i \in <S>$$
So $< S >$ is closed with respect to the operations of vector addition and scalar multiplication.

The second thing we need to check is that $< S >$ satisfies the conditions for a vector space with respect to $+$ and $\cdot$, namely that for all $u, v, w \in < S >$ and all scalars $r, s$

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. for some $0 \in V$ and all $v \in V$, $v + 0 = v$
4. $0 \cdot v = 0$
5. $1 \cdot v = v$
6. $r \cdot (u + v) = r \cdot u + r \cdot v$
7. $(r + s) \cdot v = r \cdot v + s \cdot v$
8. $(rs) \cdot v = r \cdot (s \cdot v)$

Since $V$ is a vector space these properties hold whenever $u, v, w \in V$ and for any scalars $r, s$. So these properties certainly hold when $u, v, w \in < S >$ which is a subset of $V$.

3. Let $V$ be the vector space $\mathbb{R}C[-\pi, \pi]$ of all real-valued continuous functions on $[-\pi, \pi]$, with inner product $< \cdot, \cdot >$ defined by

$$< y, z > = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x)z(x)dx$$

Prove that $< \cdot, \cdot >$ does indeed define an inner product.

We need to verify that the conditions for an inner product hold in the case of $< \cdot, \cdot >$, namely that

1. $< av + bw, u > = a < v, u > + b < w, u >$
2. $< w, v > = < v, w >$
3. $< v, v > \geq 0$
4. $< v, v > = 0$ if and only if $v = 0$. 

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for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}C[-\pi, \pi]$ and all $a, b \in \mathbb{R}$. Here condition (2) takes account of the fact that the field of scalars is $\mathbb{R}$. We check these conditions in the order they are listed.

1. $<a\mathbf{v} + b\mathbf{w}, \mathbf{u}> = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(a\mathbf{v}(x) + b\mathbf{w}(x)\right)\mathbf{u}(x)dx = \frac{a}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}(x)\mathbf{u}(x)dx + b \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{w}(x)\mathbf{u}(x)dx = a <\mathbf{v}, \mathbf{u}> + b <\mathbf{w}, \mathbf{u}>

2. $<\mathbf{w}, \mathbf{v}> = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{w}(x)\mathbf{v}(x)dx = <\mathbf{v}, \mathbf{w}>$

3. $<\mathbf{v}, \mathbf{v}> = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}(x)^2dx \geq 0$ since $\mathbf{v}(x)^2 \geq 0$ for all $x$.

4. Evidently $<\mathbf{v}, \mathbf{v}> = 0$ if $\mathbf{v} = \mathbf{0}$. We need to prove the converse, namely that if $\mathbf{v} : [-\pi, \pi] \to \mathbb{R}$ is continuous then $\mathbf{v}(x) = 0$ for all $x \in [-\pi, \pi]$.

Some care is needed with this because the condition does not hold for functions with discontinuities at single points. Here is a formal proof based on the $\epsilon - \delta$ definition of continuity.

If $\mathbf{v} \neq \mathbf{0}$ choose $x_0 \in [-\pi, \pi]$ with $\mathbf{v}(x_0) \neq 0$. If necessary replace $\mathbf{v}$ by $-\mathbf{v}$, so that $\mathbf{v}(x_0) > 0$. Then set

$$\epsilon = \frac{\mathbf{v}(x_0)}{2}$$

Because $\mathbf{v}$ is continuous at $x_0$ we can choose $\delta > 0$ with the property that

$$|\mathbf{v}(x) - \mathbf{v}(x_0)| < \epsilon$$

whenever $|x - x_0| < \delta$

Then for $x \in (x_0 - \delta, x_0 + \delta)$ we have

$$\mathbf{v}(x) > \mathbf{v}(x_0) - \epsilon = \epsilon$$

Therefore

$$<\mathbf{v}, \mathbf{v}> = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{v}(x)^2dx \geq \frac{1}{2\pi} \int_{x_0-\delta}^{x_0+\delta} \mathbf{v}(x)^2dx$$

where we interpret $\mathbf{v}(x)$ as 0 when $x$ lies outside the domain $[-\pi, \pi]$ of $\mathbf{v}$. Then the right hand side is at least as large as

$$\frac{\delta \epsilon^2}{2\pi} > 0$$

This won’t affect $<\mathbf{v}, \mathbf{v}>$ and if $-\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$ also.
since at least half of the interval \([x_0 - \delta, x_0 + \delta]\) is contained in \([-\pi, \pi]\). So \(<\mathbf{v}, \mathbf{v}>>0\) and this completes the proof.

4. In exercise 3 above, show that the set

\[ S = \{x^n : n \in \mathbb{Z}\} \]

is linearly independent.

Suppose that for some \(n \geq 0\) there are constants \(c_0, c_1, \ldots c_n\) such that

\[ f(x) = c_0 + c_1 x + c_2 x^2 + \ldots c_n x^n = 0 \]

for every \(x \in [-\pi, \pi]\). Then

\[ c_0 = f(0) = 0,\quad c_1 = f^{(1)}(0) = 0,\quad \ldots c_n = f^{(n)}(0) = 0 \]

where \(f^{(n)}\) is the \(n\)-th order derivative of \(f\). Since

\[ c_0 = c_1 = c_2 = \ldots = c_n = 0 \]

\(S\) is linearly independent.

5. In exercise 3 above, show that the set

\[ S = \{\cos(nx), \sin(nx) : n \in \mathbb{Z}_+\} \subset V \]

is linearly independent.

Suppose that for some \(n \geq 0\) there are constants \(a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n\) such that

\[ y(x) = a_1 \cos x + b_1 \sin x + a_2 \cos(2x) + b_2 \sin(2x) + \ldots + a_n \cos(nx) + b_n \sin(nx) = 0 \]

for all \(x \in [-\pi, \pi]\). Then the coefficients \(a_i, b_i\) are Fourier coefficients of \(y\). Because \(y(x) = 0\) for all \(x \in [-\pi, \pi]\) the Fourier coefficients are all 0, from the formulae proved in notes for Fourier coefficients.

6. In exercise 3 above, show that the set

\[ S = \{e^{nx} : n \in \mathbb{Z}\} \]

is linearly independent.

Suppose that for some coefficients \(c_m, c_{m+1}, \ldots c_{n-1}, c_n \in \mathbb{R}\) we have

\[ g(x) = c_m e^{mx} + c_{m+1} e^{(m+1)x} + \ldots + c_{n-1} e^{(n-1)x} + c_n e^{nx} = 0 \]

for all \(x \in [-\pi, \pi]\)
Since $e^{mx} \neq 0$ for all $x \in \mathbb{R}$ we can write

$$d_0 + d_1 e^x + \ldots + d_p e^{px} = 0 \text{ for all } x \in [-\pi, \pi]$$

where $d_0 = c_m, d_1 = c_{m+1}, \ldots, d_p = c_n$ and $p = n - m$. It suffices to prove that all the $d_j$ are 0. If they are not all 0 suppose, without loss, that $d_p \neq 0$. Substituting $z = e^x$ we have

$$h(z) = d_0 + d_1 z + \ldots + d_p z^p = 0 \text{ for all } z \in [e^{-\pi}, e^{\pi}]$$

The polynomial $f(w) \equiv h(w + 1)$ therefore satisfies

$$f(w) = 0 \text{ for all } w \text{ such that } |w| < 1 - e^{-\pi}$$

So the coefficients of $f$ are all 0, by the same argument as in Exercise 4. In particular the coefficient of $w^p$ is zero. However the coefficient of $w^p$ in $f(w)$ is $d_p$, and this is a contradiction. So all the $d_j$ are 0 after all, and $S$ is linearly independent.

7. Prove that a norm $\| \|$ on a finite-dimensional real vector space $V$ is associated with an inner product if and only if

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$$

The necessity of this condition is rather clear, since when $\| \|$ comes from an inner product $\langle \ , \ \rangle$

$$\|v + w\|^2 + \|v - w\|^2 = \langle v + w, v + w \rangle + \langle v - w, v - w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle + \langle v, v \rangle - 2\langle v, w \rangle + \langle w, w \rangle = 2(\|v\|^2 + \|w\|^2)$$

To prove the converse we define

$$\langle v, w \rangle = \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2)$$

and then check that $\langle \ , \ \rangle$ satisfies the conditions for an inner product. Evidently $\langle v, w \rangle = \langle w, v \rangle$ and $\langle v, v \rangle = \|v\|^2 \geq 0$. It follows then from the properties of the norm that $\langle v, v \rangle = 0$ if and only if $v = 0$. It remains to prove that $\langle v, w \rangle$ is linear in the variable $v$.

According to our definition of $\langle \ , \ \rangle$

$$4 \langle u + v, w \rangle = \|u + v + w\|^2 - \|u + v - w\|^2 =$$
\[\|u + v + w\|^2 + \|u - v - w\|^2 - \|u - w + v\|^2 - \|u - w - v\|^2 = 2\|u\|^2 + 2\|v + w\|^2 - 2\|u - w\|^2 - 2\|v\|^2 - 2\|w\|^2\]

where we use the given condition on \(\|\cdot\|\). Rewrite the right hand side in the form

\[2\|u\|^2 + 2\|w\|^2 + 2\|v + w\|^2 - 2\|u - w\|^2 - 2\|v\|^2 - 2\|w\|^2\]

and apply the condition again. We get

\[\|u + w\|^2 + \|u - w\|^2 + 2\|v + w\|^2 - 2\|u - w\|^2 - \|v + w\|^2 - \|v - w\|^2 = 4 < u, w > + 4 < v, w >\]

It remains to prove that \(< r.v, w > = r < v, w >\) for any \(r \in \mathbb{R}\).

If \(m\) is a positive integer then

\(< m.v, w > = < v + \ldots + v, w > = < v, w > + \ldots < v, w > = m < v, w >\)

Applying this to \(\frac{1}{n} v\) instead of \(v\), where \(n\) is any nonzero integer

\[n < \frac{m}{n}.v, w > = mn < \frac{1}{n}.v, w > = m < v, w >\]

after a second application. So

\(< q.v, w > = q < v, w >\) for any rational number \(q\)

Then for \(r \in \mathbb{R}\) choose a sequence

\[\{q_1, q_2, \ldots, q_i, \ldots\}\]

of rational numbers converging to \(r\). From the definition of \(< s.v, w >\) it is rather easy to see that \(< s.v, w >\) is continuous in \(s \in \mathbb{R}\), and so is \(s < v, w >\) of course. Therefore

\(< r.v, w > = \lim_{i \to \infty} < q_i.v, w > = \lim_{i \to \infty} q_i < v, w > = r < v, w >\)

as required.

8. Prove that a convergent sequence in a normed vector space is always Cauchy.
Suppose that \( \{s_n : n \geq 1\} \) converges, to \( s_\infty \) say. Then, given \( \epsilon > 0 \), we can choose \( N \) so large that\(^2\)

\[
\|s_n - s_\infty\| < \frac{\epsilon}{2} \text{ whenever } n \geq N
\]

Then if both \( m, n \geq N \) we have

\[
\|s_n - s_\infty\| < \frac{\epsilon}{2} \text{ and } \|s_m - s_\infty\| < \frac{\epsilon}{2}
\]

The triangle inequality then gives

\[
\|s_m - s_n\| = \|(s_m - s_\infty) - (s_n - s_\infty)\| \leq \|s_n - s_\infty\| + \|s_m - s_\infty\| < 2\frac{\epsilon}{2} = \epsilon
\]

This shows that \( \{s_n : n \geq 1\} \) is Cauchy.

9. Let \( \ell^2 \) be the set of all doubly-infinite sequences

\[
\{c_n \in \mathbb{C} : n \in \mathbb{Z}\}
\]

with the property that

\[
\Sigma_{n \in \mathbb{Z}} |c_n|^2 < \infty
\]

Prove that \( \langle , \rangle \) defined on \( \ell^2 \) by

\[
\langle c, d \rangle = \Sigma_{n \in \mathbb{Z}} c_n\bar{d}_n
\]

is an inner product.

The first thing we need to establish is that \( \langle c, d \rangle \) is defined, namely that the limit

\[
\lim_{m \to \infty} s_m
\]

exists, where

\[
s_m = \Sigma_{n=-m}^{m} c_n\bar{d}_n
\]

It is enough to consider sequences that are 0 for values of \( n \geq 0 \) or \( n < 0 \), since a doubly-infinite sequence \( c \) is the sum of two such sequences, and inner products also add. Suppose from now on that \( c_n = 0 \) for \( n < 0 \).

From the Cauchy-Schwarz inequality for the finite-dimensional inner product space \( \mathbb{C}^{n+1} \) it follows that for \( n > 0 \) we have

\[
\Sigma_{j=0}^{n}|c_j\bar{d}_j| \leq \sqrt{\Sigma_{j=0}^{n}|c_j|^2} \sqrt{\Sigma_{k=0}^{n}|d_k|^2} \leq \sqrt{\Sigma_{j=0}^{\infty}|c_j|^2} \sqrt{\Sigma_{k=0}^{\infty}|d_k|^2}
\]

\(^2\)The definition of convergence refers to \( \epsilon \), but \( \frac{\epsilon}{2} \) is also a positive number, and we’re entitled to use that if we like. In this exercise it helps.
where the series on the right converge, because \( c, d \in \ell^2 \). Therefore
\[
\sum_{j=0}^{\infty} |c_j d_j|
\]
converges, namely
\[
\sum_{j=0}^{\infty} c_j \bar{d}_j
\]
converges absolutely.
Now that we know \(<c,d>\) is defined, we can carry out the simple manipulations needed to check that \(<\cdot,\cdot>\) is an inner product. Evidently
\begin{itemize}
  \item \(<c,d>\) is the conjugate of \(<d,c>\) since the corresponding terms in the series are conjugate
  \item \(<c,c>\geq 0\) since each \(|c_n|^2 \geq 0\), and for the same reason \(<c,c> = 0\) only when every \(c_n = 0\), namely when \(c = 0\)
  \item \(<c + d,e> = <c,e> + <d,e>\) since the differences between the corresponding partial bounded by the tails of convergent series.
\end{itemize}

10. In exercise 9 prove that \(\ell^2\) is complete with respect to the inner product \(<\cdot,\cdot>\).
Let \(c = \{c^p : p \geq 1\}\) be a Cauchy sequence in \(\ell^2\) where
\[
c^p = \{c_n^p : n \in \mathbb{Z}\}
\]
Because
\[
|c_n^p - c_n^q|^2 \leq \|c^p - c^q\|^2
\]
it follows that the sequence of real numbers
\[
\{c_n^p : p \geq 1\}
\]
is also Cauchy, for every \(n \in \mathbb{Z}\). For each such \(n\) define \(c_n^\infty\) to be the limit of the Cauchy sequence \(\{c_n^p : p \geq 1\}\). Let \(c^\infty\) be the doubly-infinite sequence
\[
\{c_n^\infty : n \in \mathbb{Z}\}
\]
We have yet to show that \(c^\infty \in \ell^2\) and that \(\{c^p : p \geq 1\}\) converges to \(c^\infty\).
Given \(\epsilon > 0\) choose \(P\) so large that
\[
\|c^p - c^q\| < \epsilon \text{ whenever } p, q \geq P
\]
We can do this because \(c\) is Cauchy.
Then for each $n \in \mathbb{Z}$ choose $P_n$ so large that

$$|c_n^p - c_n^{\infty}| < \frac{\epsilon}{2|n|}$$

whenever $p \geq P_n$. We can do this because the sequences of reals converge, and without loss $P_n \geq P$ for every $n$.

For $p \geq P$ and any $n \in \mathbb{Z}$

$$|c_n^p - c_n^{\infty}| \leq |c_n^p - c_n^{P_n}| + |c_n^{P_n} - c_n^{\infty}| \leq |c_n^p - c_n^{P_n}| + \frac{\epsilon}{2|n|}$$

Notice that the first term on the right is bounded above by $\epsilon$, since $p, P_n \geq P$.

Then

$$|c_n^p - c_n^{\infty}|^2 \leq |c_n^p - c_n^{P_n}|^2 + \frac{2\epsilon^2}{2|n|} + \frac{\epsilon^2}{2^2|n|} \quad (1)$$

Therefore

$$\|c^p - c^{\infty}\|^2 \leq \epsilon^2 + \beta \epsilon^2$$

where $\beta$ is some positive constant, and so

- $c^p - c^{\infty} \in \mathbb{R}^2$, and
- $\lim_{p \to \infty} c^p = c^{\infty}$