This test has 5 questions on 2 pages. Each question is worth 2%.

The test counts for 10% towards assessment.

Time permitted: 45 minutes

1. For \( n \in \mathbb{Z} \) define \( y_n : [-\pi, \pi] \rightarrow \mathbb{C} \) by

\[
y_n(x) = e^{nix}
\]

where \( i^2 = -1 \) and \( x \in [-\pi, \pi] \). Compute the integral

\[
\int_{-\pi}^{\pi} y_n(x) y_m(x) \, dx
\]

where \( m, n \in \mathbb{Z} \).

2. Let \( y : [-\pi, \pi] \rightarrow \mathbb{C} \) be a function which can be written as an absolutely convergent series of the form

\[
y(x) = \sum_{n \in \mathbb{Z}} c_n e^{nix}
\]

Explain how the answer to question (1) above can be used to calculate the coefficients \( c_n \in \mathbb{C} \) from the function \( y \).

3. Let \( y(x) = x \) for \( x \in (-\pi, \pi) \). Find coefficients \( a_n, b_n \in \mathbb{C} \) so that

\[
y(x) = \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos(nx) + b_n \sin(nx))
\]

and show your working. Comment on what happens when you substitute \( x = \pi \) in the series for \( y(x) \) in \((-\pi, \pi)\).

4. State Parseval’s identity for a piecewise continuous function \( y : [-\pi, \pi] \rightarrow \mathbb{C} \). Suppose that the expansion

\[
x = 2(\sin x - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \ldots \ldots)
\]

is valid for \( x \in (-\pi, \pi) \). Prove that

\[
\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots
\]
5. Let $V$ be the vector space of continuous functions $y : [-\pi, \pi] \to \mathbb{R}$ with the usual vector space operations of pointwise addition of functions and pointwise multiplication of functions by scalars in $\mathbb{R}$. Let $\langle \ , \ \rangle$ be the inner product on $V$ given by

$$\langle y, z \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) z(x) \, dx$$

Is $V$ complete with respect to the norm defined by this inner product? Give your reasoning.

Hint: Consider the sequence $\{s_n : n \geq 1\} \subset V$ of odd functions $s_n$ given by

$$s_n(x) = nx \text{ for } 0 \leq x \leq \frac{1}{n} \text{ and } s_n(x) = 1 \text{ for } x > \frac{1}{n}$$

and define $s_\infty : [-\pi, \pi] \to \mathbb{R}$ to be the odd function given by

$$s_\infty(x) = 1 \text{ for } x > 0.$$

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