2CA1 Solutions to Short Test 1

1. For \( n \in \mathbb{Z} \) define \( y_n : [-\pi, \pi] \to \mathbb{C} \) by

\[ y_n(x) = e^{nix} \]

where \( i^2 = -1 \) and \( x \in [-\pi, \pi] \). Compute the integral

\[ \int_{-\pi}^{\pi} y_n(x)\bar{y}_m(x)\,dx \]

where \( m, n \in \mathbb{Z} \).

\[ \int_{-\pi}^{\pi} y_n(x)\bar{y}_m(x)\,dx = \int_{-\pi}^{\pi} e^{(n-m)ix}\,dx \quad (1) \]

- if \( n = m \) the right hand side of (1) is
  \[ \int_{-\pi}^{\pi} 1\,dx = 2\pi \]

- if \( n \neq m \) the right hand side is
  \[ \frac{1}{(n-m)i} \left[ e^{(n-m)ix}\right]_{-\pi}^{\pi} = \frac{1}{(n-m)i}((-1)^{n-m} - (-1)^{m-m}) = 0 \]

So

\[ \int_{-\pi}^{\pi} y_n(x)\bar{y}_m(x)\,dx = 2\pi\delta^n_m \]

where \( \delta^n_m \) is 1 or 0 according as \( n = m \) or not.

2. Let \( y : [-\pi, \pi] \to \mathbb{C} \) be a function which can be written as an absolutely convergent series of the form

\[ y(x) = \sum_{n \in \mathbb{Z}} c_n e^{nix} \]

Explain how the answer to question (1) above can be used to calculate the coefficients \( c_n \in \mathbb{C} \) from the function \( y \). Substituting \( y(x) \) in the answer\(^1\), we obtain

\[ \int_{-\pi}^{\pi} y(x)\bar{y}_m(x)\,dx = \sum_{n \in \mathbb{Z}} c_n \int_{-\pi}^{\pi} \bar{y}_m(x)\,dx = 2\pi \sum_{n \in \mathbb{Z}} c_n \delta^n_m = 2\pi c_m \]

\(^1\)You are expected to know the answer from notes, even if you have forgotten how to compute it.
Therefore

\[ c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \tilde{g}_m(x) \, dx \]

and we can replace \( m \) by \( n \) on both sides of this formula to get

\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \tilde{g}_n(x) \, dx \]

3. Let \( y(x) = x \) for \( x \in (-\pi, \pi) \). Find coefficients \( a_n, b_n \in \mathbb{C} \) so that

\[ y(x) = \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos(nx) + b_n \sin(nx)) \]

and show your working. Comment on what happens when you substitute \( x = \pi \) in the series for \( y(x) \) in \((-\pi, \pi)\).

Because \( y \) is an odd function \( a_n = 0 \) for \( n \geq 0 \). For \( n \geq 1 \)

\[ b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) \, dx = \frac{2}{\pi} \left( \frac{x - \cos(nx)}{n} \right)_{0}^{\pi} + \frac{1}{n} \int_{0}^{\pi} \cos(nx) \, dx \]

on integration by parts. The right hand side simplifies to

\[ \frac{2}{\pi} \left( \frac{-(-1)^n\pi}{n} + \frac{1}{n} \left[ \sin(nx) \right]_{0}^{\pi} \right) = \frac{-2(-1)^n}{n} \]

Therefore

\[ y(x) = 2(\sin x - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \frac{1}{4} \sin(4x) + \ldots \ldots) \]

Substituting \( x = \pi \) into the series, we obtain 0 and yet \( y(\pi) = \pi \neq 0 \). This does not contradict the series expansion for \( y(x) \) when \( x \in (-\pi, \pi) \), nor even the Dirichlet-Jordan theorem when \( x = \pi \). To apply the Dirichlet-Jordan theorem in this case \( y \) needs to be turned into a periodic function defined on the whole of \( \mathbb{R} \) and then the theorem says

\[ \frac{1}{2} \left( \lim_{x \to \pi^-} y(x) + \lim_{x \to \pi^+} y(x) \right) = 0 \]

Since \( y \) is extended periodically the left hand side is

\[ \frac{1}{2} \left( \lim_{x \to \pi^-} y(x) + \lim_{x \to \pi^+} y(x) \right) = \frac{1}{2}(\pi - \pi) = 0 \]

which agrees with the Dirichlet-Jordan theorem.
4. State Parseval’s identity for a piecewise continuous function $y : [-\pi, \pi] \to \mathbb{C}$. Suppose that the expansion

$$x = 2(\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \ldots)$$

is valid for $x \in (-\pi, \pi)$. Prove that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

If the exponential series expansions for $y(x), z(x)$ are

$$\sum_{n \in \mathbb{Z}} c_n e^{inx} \text{ and } \sum_{n \in \mathbb{Z}} d_n e^{inx}$$

respectively then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \bar{z}(x) dx = \sum_{n \in \mathbb{Z}} c_n \bar{d}_n$$

In this case, taking $y(x) = z(x) = x$ we have

$$\frac{\pi^2}{3} = \sum_{n \in \mathbb{Z}} |c_n|^2$$

where $c_0 = 0, c_n = \frac{(-1)^n i}{n}$. Therefore

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

5. Let $V$ be the vector space of continuous functions $y : [-\pi, \pi] \to \mathbb{R}$ with the usual vector space operations of pointwise addition of functions and pointwise multiplication of functions by scalars in $\mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $V$ given by

$$\langle y, z \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \bar{z}(x) dx$$

Is $V$ complete with respect to the norm defined by this inner product? Give your reasoning.

Consider the sequence $\{s_n : n \geq 1\} \subset V$ of odd functions $s_n$ given by

$$s_n(x) = nx \text{ for } 0 \leq x \leq \frac{1}{n} \text{ and } s_n(x) = 1 \text{ for } x > \frac{1}{n}$$
and define $s_\infty : [-\pi, \pi] \to \mathbb{R}$ to be the odd function given by

$$s_\infty(x) = 1 \text{ for } x > 0$$

Then $s_\infty \notin V$ and yet $s_\infty$ does belong to the larger vector space $W$ of piecewise-continuous functions on $[-\pi, \pi]$ with a finite number of finite jump discontinuities.

The formula for $< , >$ also defines an inner product on $W$, provided we identify functions which disagree only on a finite set of points. If we can show that

$$s_n \to s_\infty$$

then it follows that $\{s_n : n \geq 1\}$ is Cauchy, even as a sequence in $V$. However if $\{s_n : n \geq 1\}$ does not converge to an element of $V$ because there is no continuous function which disagrees with $s_\infty$ only on a finite number of points.

It remains to show that $s_n \to s_\infty$ in $W$. We have

$$\|s_\infty - s_n\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (s_\infty(x) - s_n(x))^2 \, dx =$$

$$\frac{1}{\pi} \int_{0}^{\pi} (s_\infty(x) - s_n(x))^2 \, dx \leq$$

$$\frac{1}{\pi} \int_{0}^{\pi} 1 \, dx = \frac{1}{n\pi}$$

and we can make this as small as we like by taking $n$ large.