Permutation groups and normal subgroups

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Basic permutation groups

Finite group: $\leftrightarrow$ set of finite simple groups (composition factors)

Symmetric group on $\Omega$: $\mathrm{Sym}(\Omega) = \{\text{all permutations of } \Omega\}$.

Permutation group on $\Omega$: $G \leq \mathrm{Sym}(\Omega)$; called finite if $\Omega$ finite.

Finite permutation group: $\leftrightarrow$ set of ‘basic’ permutation groups

Aim: to describe

- various ‘basic’ permutation groups
- their structure, and applications in graph theory
Outline of lecture

• basic permutation groups: primitive and quasiprimitive groups
• edge-transitive graphs
• structure of basic groups: O’Nan-Scott types
• applications to groups and graphs
• simple groups and basic permutation groups
Transitive permutation groups

Permutation group $G$ on $\Omega$: **transitive** $\iff \forall \alpha, \beta \in \Omega, \exists g \in G$ such that $\alpha^g = \beta$

Let $\alpha \in \Omega$. Set $H = G_\alpha = \text{stabiliser in } G \text{ of } \alpha$.

Suppose $H < K < G$.

Set $B = \{K-\text{images of } \alpha\}$, $\mathcal{P} = \{G-\text{images of } B\}$.
Then $\mathcal{P}$ is a partition of $\Omega$.

**Induces two smaller transitive groups:**

$G_1 = K^B$ on $B$ and $G_2 = G^\mathcal{P}$ on $\mathcal{P}$. 
Basic permutation groups I

Subgroup chain: \( H < K_1 < \ldots < K_k = G \) gives \( k \) smaller transitive groups \( G_1, \ldots, G_k \), and an embedding

\[
G \leq G_1 \wr G_2 \wr \ldots \wr G_k < \text{Sym}(\Omega) \quad \text{wreath product}
\]

First type: \( G \) primitive \( \iff \) \( H \) maximal in \( G \)

Choose maximal chain: then each \( G_i \) is primitive

Second type: \( G \) quasiprimitive \( \iff \) each nontrivial normal subgroup \( N \lhd G \) transitive

Choose chain maximal such that: each \( K_i = HN_i \) with \( N_i \lhd K_{i+1} \). Then each \( G_i \) is quasiprimitive.
Basic permutation groups II

**Finite primitive groups:** Traditional choice for basic groups. Of importance since beginnings of Group Theory.

**Finite quasiprimitive groups:** Named by Wielandt in 1970's. Importance emerged from combinatorial applications in 1980's.

**primitive \(\Rightarrow\) quasiprimitive:** by definition.

**quasiprimitive \(\not\Rightarrow\) primitive:** for example, if \(G\) nonabelian simple and \(H\) not maximal then \(G\) is quasiprimitive but not primitive.
Graphs I

\[ \Gamma = (\Omega, E) : \quad \Omega = \text{vertex set}; \quad E = \text{edge set}. \]

**Automorphism of** \( \Gamma \): edge-preserving permutation of \( \Omega \)

**Automorphism group of** \( \Gamma \): \( \text{Aut}(\Gamma) \leq \text{Sym}(\Omega) \).

\[ G \leq \text{Aut}(\Gamma) : \quad G \text{ transitive on } \Omega \iff G\text{-vertex-transitive} \]
\[ G \text{ transitive on } E \iff G\text{-edge-transitive} \]

**Among finite edge-transitive graphs**: are there some that we might regard as ‘basic’?
Graphs II: quotient graphs

Edge-transitive graph $\Gamma$: $G \leq \text{Aut}(\Gamma)$, $G$ transitive on $\Omega, E$. Suppose $G$ not primitive on $\Omega$. Let $\alpha \in \Omega$ and $H = G_\alpha$.

Then exists: $H < K < G$; $B = \{K-\text{images of } \alpha\}$, corresponding partition $\mathcal{P}$.

Quotient graph: $\Gamma_\mathcal{P} = (\mathcal{P}, E_\mathcal{P})$ where $C, C' \in \mathcal{P}$ adjacent $\iff$ some $\beta \in C, \beta' \in C'$ are adjacent in $\Gamma$.

$G$ transitive: on $\mathcal{P}, E_\mathcal{P}$ so $\Gamma_\mathcal{P}$ is smaller edge-transitive graph.

$\Gamma$ connected $\Rightarrow \Gamma_\mathcal{P}$ connected; then $B$ contains no edges of $\Gamma$. 
Basic Graphs: two types

Suppose $\Gamma$ connected and $G$-edge-transitive:

$K$ maximal in $G$: $G$ primitive and edge-transitive on $\Gamma_P$.

$K = HN$ with $N$ maximal such that $N \triangleleft G$ and $N$ intransitive:

$P = \{N\text{-orbits in } \Omega\}$ and $G$ quasiprimitive and edge-transitive on $\Gamma_P$; here $\Gamma_P$ called a normal quotient.

Special case: $|G : K| = |P| = 2 \Rightarrow \Gamma$ bipartite and $\Gamma_P = K_2$.

Note: Some important graph properties not inherited by primitive edge-transitive quotients; but are inherited by normal quotients.
**Structure of basic groups: O’Nan-Scott types**

**O’Nan-Scott Theorem 1979:** Finite primitive $G$ on $\Omega \Rightarrow 8$ disjoint types.

**CEP 1993:** Finite quasiprimitive $G$ on $\Omega \Rightarrow 8$ disjoint types.

**Type descriptions:** in terms of socle $\text{soc} (G) = \text{product of minimal normal subgroups}$.

**Examples:**
- **AS - almost simple type:** $\text{soc} (G) = T \triangleleft G \leq \text{Aut}(T)$, $T$ nonabelian simple group.
- **HA - affine type:** $\text{soc} (G) = Z_p^d < G \leq \text{AGL} (d, p)$
Maximal subgroups of $A_n$ and $S_n$

**Idea:** Find maximal subgroup $M$ of each type; decide if $M$ maximal in $S_n$, and if $M \cap A_n$ maximal in $A_n$; eg maximal intransitive $M = S_k \times S_{n-k}$ for $1 \leq k \leq n/2$ and $M$ is maximal in $S_n$ unless $k = n/2$.

**Liebeck P Saxl 1987:** $\exists$ subgroup list $\mathcal{L}$ such that

(a) $M$ maximal in $A_n$ or $S_n \Rightarrow M \in \mathcal{L}$ or $M$ almost simple and primitive

(b) $H$ almost simple and primitive and not maximal in $A_n$ or $S_n$$\Rightarrow H, n$ known

**Comments:**

(a) explicit list of primitive almost simple subgroups infeasible

(b) several earlier partial results

(c) depends on finite simple group classification, especially max factns
Maximal factorisations

$G$ almost simple: $T \trianglelefteq G \leq \text{Aut}(T)$

Maximal factorisation of $G$: $G = AB$, $A$ and $B$ maximal, $T \not	rianglelefteq A, B$.

Hering Liebeck Saxl 1986: Classify for $T$ exceptional Lie type

Liebeck P Saxl 1990: Classify all others - especially classical groups $G$. 
Distance transitive graphs

\[ \Gamma = (\Omega, E) : \text{ for } 0 \leq i \leq \text{diameter}, \Gamma_i = \{(\alpha, \beta) | \text{distance}(\alpha, \beta) = i\} \]

\( G \text{ distance transitive on } \Gamma : G \text{ transitive on each } \Gamma_i \)

Let \( \alpha \in \Omega, H = G_\alpha \). Suppose \( H < K < G \); \( B = K \)-orbit containing \( \alpha \), corresponding partition \( \mathcal{P} \).

\[ |G : K| > 2: \Rightarrow G \text{ distance transitive (d.t.) on } \Gamma_{\mathcal{P}} = (\mathcal{P}, E_{\mathcal{P}}) \]

\[ |G : K| = 2: \Rightarrow \Gamma \text{ bipartite, } \mathcal{P} = \{\Omega_1, \Omega_2\}, \text{ and } \Gamma_1 = (\Omega_1, E_1) \text{ distance transitive where } \{\alpha, \beta\} \in E_1 \Leftrightarrow \text{distance}(\alpha, \beta) = 2. \]
Finite primitive distance transitive graphs

Each finite d.t.g. $\Gamma$: leads to primitive d.t.g.

Using O’Nan–Scott Theorem

P, Saxl & Yokoyama 1987: $G$ primitive and d.t. on $\Gamma \Rightarrow \Gamma$ known or $G$ affine type or almost simple type.

Huge effort by many researchers.

Finite primitive d.t.g.: almost classified
Finite $s$-arc transitive graphs

For $s \geq 1$, $s$-arc in $\Gamma$: vertex sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$, such that $
abla_{i-1} \neq \alpha_{i+1}$, and $\{\alpha_i, \alpha_{i+1}\}$ an edge.

Let $\alpha \in \Omega$, $H = G_\alpha$; suppose $H < K < G$; let $\mathcal{P}$ be corresponding partition.

$G$ $s$-arc transitive on $\Gamma$ ($s \geq 2$): $\nabla G$ $s$-arc transitive on $\Gamma_\mathcal{P}$.

BUT if $K = NH$ with $N \triangleleft G$ then $\mathcal{P} = \text{set of } N\text{-orbits}$ and $G$ is $s$-arc transitive on $\Gamma_\mathcal{P}$; and if $|\mathcal{P}| > 2$ then $\Gamma$ covers $\Gamma_\mathcal{P}$.

Choose $N$ maximal intransitive: then $G$ is $s$-arc transitive and quasiprimitive on $\Gamma_\mathcal{P}$, or $|\mathcal{P}| = 2$ and $\Gamma$ bipartite.
Quasiprimitive $s$-arc transitive graphs

**P 1993:** $G$ $s$-arc transitive and quasiprimitive on $\Gamma \Rightarrow G$ is one of 4 of the possible 8 O’Nan–Scott types.

- affine type classified (Ivanov & P)
- almost simple classifications for some classes of small rank almost simple groups (Fang, Hassani, Nochefranca, Wang, P)
- twisted wreath good description (Baddeley)
- product action examples, constructions

**Li 2001:** $s \geq 4 \Rightarrow$ number of vertices odd and not a power of 2.
Over-groups of $G \leq \text{Sym}(\Omega)$

$H$ is over-group of $G$: $G < H \leq \text{Sym}(\Omega)$

Importance:

1. Test maximality of $G$.
2. Identify graph $\Gamma$, given $G \leq \text{Aut}(\Gamma)$, may need to find $\text{Aut}(\Gamma)$
3. Analyse classes of graphs $\Gamma$ constructed using some $G \leq \text{Aut}(\Gamma)$

Note:

$G$ primitive $\Rightarrow$ $H$ primitive
$G$ quasiprimitive $\not\Rightarrow$ $H$ quasiprimitive
but knowing the quasiprimitive over-groups $H$ is crucial
Over-group classifications: \(G < H \leq \text{Sym}(\Omega)\)

Liebeck, Saxl & P 1987: \(G, H\) both almost simple, primitive, and \(G\) maximal in \(H\).

P 1990: all other types with \(G, H\) primitive


Incomplete information if: \(H\) in product action or if \(G, H\) both almost simple.
Applications of over-group classifications

Analysing classes of graphs: distinguishing similar graphs in a class; establishing properties etc

Li 2001: explicit classification of primitive and bi-primitive $s$-arc transitive graphs with $s \geq 4$

Liebeck, Saxl & P 2002: $G$ primitive and edge-transitive on $\Gamma \Rightarrow G, \text{Aut}(\Gamma)$ have same socle, or $\exists G < H \leq \text{Aut}(\Gamma)$ with $\text{soc}(G) \neq \text{soc}(H)$ and $G, H$ in explicit list
Subgroups of finite simple groups

(New results: apart from maximal subgroups of $A_n, S_n$; maximal factorisations of almost simple groups.)

$\Pi(T)$: set of prime divisors of $|T|$

Liebeck, Saxl & P 2001: Discovered that, for $T$ simple, there is a small subset of $\Pi(T)$ that is rarely present in proper subgroups.

Given $T$ simple: $\exists$ explicit subset $\Pi \subseteq \Pi(T)$ such that $|\Pi| \leq 3$ and if $M < T$ with $\Pi \subseteq \Pi(M)$ then $M, T$ known.
Subgroups of finite simple groups II

\( \forall \ n \geq 5, \ A_n \text{ simple, } A_{n-1} \text{ maximal subgroup}, \ |A_n : A_{n-1}| = n. \)

\[ P_{\text{max}}(x) := \text{proportion of } n \leq x \text{ such that } n = |T : M| \text{ where } \]
\( T \text{ simple, } M < T \text{ maximal, } (T, M) \neq (A_n, A_{n-1}). \)

\[ P_{\text{proper}}(x) := \text{proportion of } n \leq x \text{ such that } n = |T : M| \text{ where } \]
\( T \text{ simple, } M < T, (T, M) \neq (A_n, A_{n-1}). \)

**Cameron, Neumann & Teague 1982:** \( P_{\text{max}} \leq \frac{(1+o(1))}{\log x}. \)

**Shalev & P 2002:** \( P_{\text{proper}} \leq \frac{(1+o(1))c}{\log x} \) where \( c = \sum_{d=1}^{\infty} \frac{1}{\phi(d)}. \)

Note 2.2 \( < c < 2.23. \)
Degrees of primitive and quasiprimitive groups

Given $G \leq \text{Sym}(\Omega)$: degree of $G := |\Omega|$

$\forall n \geq 3$: $S_n$ and $A_n$ are primitive of degree $n$

Proportion of $n \leq x$: $\exists$ primitive or quasiprimitive proper subgroup $G$ of $S_n$, with $G \neq A_n$, is at most

$$\frac{(1+o(1))2}{\log x}$$ for primitive groups (CNT 1982)

$$\frac{(1+o(1))(c+1)}{\log x}$$ for quasiprimitive groups (PS 2002)
Summary

• Basic permutation groups: primitive groups, quasiprimitive groups
• Possible structures understood via ‘O’Nan–Scott’ type theorems.
• Approach enables application of simple group classification.
• Provide ideal framework for analysing finite combinatorial structures.
• Power of results due to use of finite simple group classification.