1. The following matrix $H$ is an incomplete parity check matrix for the binary Hamming code $C$ of order 4.

$$H = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}.$$

(a) Find the missing columns $h_3, h_7, h_{12}$ and $h_{14}$.
(b) Find the parameters $n, k, d$ for $C$.
(c) Decode the following received vectors
   i. 011001001011101
   ii. 000110110000010.
(d) Suppose that a codeword was sent and that a vector $r$ was received, with $Hr^\top = h_i$ (where $h_i$ is the binary expansion for $i$). Explain why the set of all possibilities for $r$ is
   $$C + e_i = \{c + e_i \mid c \in C\}$$
   (where $e_i$ is the standard basis vector with a 1 in the $i^{th}$ position and 0s elsewhere).
(e) Prove that $(C + e_i) \cap (C + e_j) = \emptyset$ if and only if $i \neq j$.

2. Let $C_1$ be a binary linear $(n_1, k_1, d_1)$-code and $C_2$ a binary linear $(n_2, k_2, d_2)$-code, and define
   $$C_1 \times C_2 = \{c_1 c_2 \mid c_1 \in C_1, c_2 \in C_2\}$$
   (where $c_1 c_2$ denotes concatenation of the codewords $c_1$ and $c_2$).
(a) Prove that $C_1 \times C_2$ is a binary linear code.

(b) Find the parameters $n, k, d$ for $C_1 \times C_2$ (in terms of the parameters of $C_1$ and $C_2$). Give reasons for your answers.

(c) Prove that $(C_1 \times C_2)^\top = C_1^\top \times C_2^\top$.

3. The accuracy of a radio station’s weather man at predicting rain is as follows:

10% of the time he predicts rain and rain occurs;

20% of the time he predicts rain and rain does not occur;

10% of the time he predicts no rain and rain occurs; and

60% of the time he predicts no rain and no rain occurs.

So he is correct 70% of the time. A listener observes that he could be correct 80% of the time by simply always predicting no rain. He applies for the weatherman’s job. However the station manager declines to hire the listener. Why?

4. A source $S$ has message alphabet $M = \{a, b, c, d, e, f, g\}$ with probabilities given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.013</td>
<td>0.015</td>
<td>0.3</td>
<td>0.171</td>
<td>0.25</td>
<td>0.101</td>
<td>0.15</td>
</tr>
</tbody>
</table>

(a) Find a binary Huffman encoding for this source.

(b) What is the average codeword length?

(c) What is the entropy of the source?

5. What is the capacity of a binary symmetric channel with crossover probability $p = 0.01$? Is it possible to have error-free transmission over such a channel of a rate of 95%? Why?

6. For each of the comparisons below, indicate which event or message has the most information, and what leads you to this conclusion.
(a) Comparison 1:
* A sequence of 10 tosses of a fair coin.
* A sequence of 10 tosses of an unfair coin that comes up heads 1/8 of the time and tails 7/8 of the time.

(b) Comparison 2:
* A single throw of a fair 20-sided die.
* A single throw of a fair 12-sided die.
* Two throws of a fair 8-sided die.

(c) Comparison 3:
* A password containing 8 characters from lower-case letters only (assume each character is equally probable).
* A password containing 8 characters, lower-case and digits (assume each character is equally probable).
* A password containing 12 characters from lower-case letters only.

(d) Comparison 4:
* A form with four check-boxes. Assume all choices are equally likely.
* A form with two menus, each with four choices. Assume all choices are equally likely.

7. **Bonus Question**
A game involves one player A selecting an integer \( n \) between 1 and 1000 (inclusive) and the other player B trying to determine \( n \) by asking yes/no questions (such as “Is it 3?” and “Is it bigger than 100?”), which player A must answer truthfully. The aim is to determine \( n \) in the smallest number of questions. Describe a strategy by which any choice of number can be determined (on average) in the minimum number \( m \) of questions. What is \( m \)? Justify your answers.