1. Find all the subgroups of $\mathbb{Z}_{12}$. There are six of them, namely $\{0\} = \langle 0 \rangle$, $\{0, 4, 8\} = \langle 4 \rangle$, $\{0, 2, 4, 6, 8, 10\} = \langle 2 \rangle$, and $\mathbb{Z}_{12}$ itself. This can be proved by showing (as in lectures), that a subgroup of a cyclic group is cyclic, and observing that $\langle 3 \rangle = \langle 9 \rangle = \langle 4 \rangle = \langle 8 \rangle$, $\langle 2 \rangle = \langle 10 \rangle$, and $\mathbb{Z}_{12} = \langle 1 \rangle = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle$. Of course, all of these are subgroups, since, from lectures, $\langle x \rangle$ is a subgroup of $G$ for any element $x$ of $G$.

2. Show that $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\}$ is a subgroup of $GL(2, \mathbb{R})$. It is a subset, as the determinant of such a matrix is $ad$, which is not zero. It contains the identity matrix, so it's nonempty. The product of two elements $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, $\begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ of $H$ is $\begin{bmatrix} aa' & a'b' + bd' \\ 0 & dd' \end{bmatrix}$, which is an element of $H$, so $H$ is closed under matrix multiplication, the operation in the group $GL(2)$. The inverse of the element $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ of $H$ is $\begin{bmatrix} \frac{1}{d} & -\frac{b}{d^2} \\ 0 & \frac{1}{d} \end{bmatrix}$, which is an element of $H$, so $H$ is closed under inverses. Hence $H$ is a subgroup of $GL(2)$.

3. Show that, if $H$ and $K$ are subgroups of the group $G$, then $H \cap K$ is a subgroup of $G$. $e \in H$ and $e \in K$ (as $H$ and $K$ are subgroups of $G$), so $e \in H \cap K$, so $H \cap K$ is non-empty. Clearly, $H \cap K$ is a subset of $G$, as $H$ and $K$ are subsets of $G$. Let $x \in H \cap K$. Then $x \in H$, so $x^{-1} \in H$, as $H$ is closed under inverses, and $x \in K$, so $x^{-1} \in K$, as $K$ is closed under inverses. Thus $x^{-1} \in H \cap K$, giving $H \cap K$ closed under inverses. If, in addition $y \in H \cap K$, then, $xy \in H$, as $H$ is closed under multiplication, and $xy \in K$, as $K$ is closed under multiplication, giving $xy \in H \cap K$. Thus $H \cap K$ is closed under multiplication. So $H \cap K$ is a subgroup of $G$.

What is the intersection of the subgroups $2\mathbb{Z}$ and $3\mathbb{Z}$ of $\mathbb{Z}$? It is the set of integers that are both multiples of 2 and multiples of 3, that is, are multiples of 6. That is $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$.

4. **Bonus question**

Do the one-to-one continuous functions from $\mathbb{R}$ onto $\mathbb{R}$ form a subgroup of $Sym(\mathbb{R})$? Yes, they do. We know already that $Sym(\mathbb{R})$ is a group under composition, so the question reduces to whether compositions of continuous functions are continuous and the inverse of any continuous one to one function from $\mathbb{R}$ onto $\mathbb{R}$ is continuous. The first follows from the chain rule for limits. So the crux of the matter is the inverse function theorem. Here a citation is sufficient, such as Salas, Hille and Anderson, Theorem B.3.1, or you can give a proof. The standard proof goes as follows:
Step 1: A continuous one-to-one function $f$ from $\mathbb{R}$ onto $\mathbb{R}$ is either increasing or decreasing.
Suppose not. Then there exist real numbers $a, b$ and $c$ with $a < b < c$ and $f(b) < f(a) < f(c)$ (or $f(b) > f(a) > f(c)$). By the intermediate value theorem, there exists $d$ with $b < d < c$ and $f(d) = f(a)$, contrary to $f$ being one to one.

Step 2: The inverse of an increasing or decreasing function $f$ from $\mathbb{R}$ onto $\mathbb{R}$ is continuous.
Suppose that $f$ is increasing. Let $c \in \mathbb{R}$. Since $f$ is onto, there exists $b \in \mathbb{R}$ with $f(b) = c$. Let $\epsilon > 0$ be arbitrary and choose $\delta > 0$ with $f(b - \epsilon) < c - \delta$ and $c + \delta < f(b + \epsilon)$. Then, if $c - \delta < x < c + \delta$, we have $f(b - \epsilon) < x < f(b + \epsilon)$, and since $f^{-1}$ is also increasing, it follows that $b - \epsilon < f^{-1}(x) < b + \epsilon$, i.e., $f^{-1}(c) - \epsilon < f^{-1}(x) < f^{-1}(c) + \epsilon$, which shows that $f^{-1}$ is continuous at $c$.
The case where $f$ is decreasing is similar.