1. Show that in a group $G$, $(x^{-1})^{-1} = x$, for all $x \in G$.
   We know that $x \ast x^{-1} = e = x^{-1} \ast x$, by the definition of inverse. We also know that if $y \ast z = e = z \ast y$ then $y = z^{-1}$, by the uniqueness of inverse. Applying this with $y = x$ and $z = x^{-1}$, it follows that $x = (x^{-1})^{-1}$.

2. Show that in a group $G$ with operation $\ast$ and identity element $e$, if $x \ast y = e$ then $y \ast x = e$.
   Multiplying both sides of $x \ast y = e$ on the right by $x$ gives $x \ast y \ast x = e \ast x = x$, since $e$ is the identity. Multiplying this on the left by $x^{-1}$ gives $x^{-1} \ast (x \ast y \ast x) = x^{-1} \ast x$, so $(x^{-1} \ast x) \ast y \ast x = e$ so $e \ast (y \ast x) = e$ so $y \ast x = e$.

3. Show that in a group $G$ with operation $\ast$ and identity element $e$, $e^{-1} = e$.
   We know that $e \ast e = e$, by the definition of identity. We also know that if $y \ast z = e = z \ast y$ then $y = z^{-1}$, by the uniqueness of inverse. Applying this with $y = e$ and $z = e$, it follows that $e^{-1} = e$.

4. Show that the set of all $n$ by $n$ matrices over the real numbers is a group under matrix addition. What is the identity?
   The sum of two $n$ by $n$ matrices over the real numbers is an $n$ by $n$ matrix over the real numbers, so matrix addition is a binary operation on the set of all $n$ by $n$ matrices over the real numbers. Matrix addition is associative. Adding any $n$ by $n$ matrix to the $n$ by $n$ zero matrix in either order gives that matrix back again, so there is an identity element. The sum of a matrix and its negative (in either order) is the $n$ by $n$ zero matrix. Hence the set of all $n$ by $n$ matrices over the real numbers is a group under matrix addition, with identity the $n$ by $n$ zero matrix.

5. Show that $\mathbb{R}^n$ is a group under addition. Sum of two elements of $\mathbb{R}^n$ is an element of $\mathbb{R}^n$, so addition is a binary operation on $\mathbb{R}^n$. Addition of vectors is associative. Adding any vector to the zero vector in either order gives that vector back again, so there is an identity element. The sum of a vector and its negative (in either order) is the zero vector. Hence the set of all $\mathbb{R}^n$ is a group under addition.

6. Show that $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ is not a group under multiplication.
   Inverses fail to exist. For example, there is no integer $n$ such that
\[ 2 \]
\[ 2 \times n = 1. \]