Assignment 3.

1. Find all irreducible quadratic polynomials of $\mathbb{Z}_2[x]$.

2. Use your answer in Question 1 to find an irreducible polynomial of degree 4 in $\mathbb{Z}_2[x]$.

3. Explain how your answer in Question 2 can be used to construct a field with 16 elements.

4. Give addition and multiplication tables for a field with 16 elements.

5. Let $F$ be a field with $p^2$ elements, $p$ an odd prime.
   
   (a) By the last assignment, $F$ has a subring $R$ which is isomorphic to $\mathbb{Z}_p$. Show that every element of $R$ is a square in $F$.
   
   (b) Deduce that every quadratic polynomial in $R[x]$ has zeros in $F$.
   
   (c) Suppose $p(x)$ is an irreducible quadratic polynomial in $R[x]$. Deduce that $F$ is isomorphic to $R/(p(x))$.
   
   (d) Deduce that any other field with $p^2$ elements is isomorphic to $F$. 