Solutions to Exercise Sheet.

1. (a) 2,4,5,6,8,10,12,14,15,16,18.
   
   (b) Let $a \in \mathbb{Z}_n, a \neq 0$. If $(a, n) = 1$, then $a$ is a unit in $\mathbb{Z}_n$. If $(a, n) = d \neq 1$, then $n/d \neq 0$ and $a(n/d) = 0$ in $\mathbb{Z}_n$, so $a$ is a divisor of zero.

2. If $a^n = 0$, then $(1 - a)(1 + a + a^2 + \cdots + a^{n-1}) = 1 - a^n = 1$, so $1 - a$ is a unit.

3. Let $a \in R \setminus \{0\}$ and define $f, g \in R^R$ by

   $$f(r) = \begin{cases} 
   0 & \text{if } r = 0 \\
   a & \text{if } r \neq 0
   \end{cases}, \quad g(r) = \begin{cases} 
   a & \text{if } r = 0 \\
   0 & \text{if } r \neq 0
   \end{cases}.$$ 

   Then $f, g \neq 0$ and $f \cdot g = 0$, so $f$ and $g$ are divisors of zero.

4. (a) $(2X + 1)^2 = 4X^2 + 4X + 1 = 1$ in $\mathbb{Z}_4[X]$, so $2X + 1$ is a unit.
   
   (b) and (c). No, because $\mathbb{Z}$ and $\mathbb{Z}_p$ are integral domains, so the only units are the constants $\neq 0$.

5. (a) $(r, s) \in (R \times S)^* \iff \exists r' \in R, s' \in S$ such that $(r, s)(r', s') = (1, 1)$

   $\iff \exists r' \in R, s' \in S$ such that $rr' = 1$ and $ss' = 1$

   $\iff r \in R^*$ and $s \in S^* \iff (r, s) \in R^* \times S^*$. Hence $(R \times S)^* = R^* \times S^*$.

   (b) If $u \in R^*$, then $\exists u' \in R$ such that $uu' = 1$, and hence $(u + I)(u' + I) = 1 + I$, which is the identity of $R/I$. Hence $u + I$ is a unit in $R/I$.

   The converse is false. For example, if $R = \mathbb{Z}$ and $I = 3\mathbb{Z}$, then $2 + 3\mathbb{Z}$ is a unit in $\mathbb{Z}/3\mathbb{Z}$, but 2 is not a unit in $\mathbb{Z}$.

6. (a) Let $p$ be a prime. Then $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$ is a field but $\mathbb{Z}$ is not a field.
(b) Yes. \(\mathbb{Z} \times \mathbb{Z}\) is has divisors of zero, and \(I = \{0\} \times \mathbb{Z} \trianglelefteq \mathbb{Z} \times \mathbb{Z}\). Then 
\((\mathbb{Z} \times \mathbb{Z})/I \cong \mathbb{Z}\) which is an ID.

(c) Yes. \(\mathbb{Z}\) is an ID and \(\mathbb{Z}/6\mathbb{Z}\) has divisors of zero.

7. (a) \(\theta(x) = \theta(y) \Rightarrow ax = ay \Rightarrow x = y\) since \(a \neq 0\) and the cancellation law holds in \(D\). Hence \(\theta\) is one-to-one.

(b) Now suppose \(D\) is finite. Then \(\theta\) will map \(D\) onto \(D\). In particular, 
\(\theta(x) = 1\) for some \(x \in D\). But then \(ax = 1\) in \(D\), so \(a\) is a unit. 
Since \(a\) was an arbitrary non-zero element of \(D\), it follows that \(D\) is a field.

8. If \((m, n) = 1\), then \(\exists x, y \in \mathbb{Z}\) such that \(mx + ny = 1\). Then 
\[a = a^{mx+ny} = (a^m)^x(a^n)^y = (b^m)^x(b^n)^y = b^{mx+ny} = b.\]

9. (a) \(a^2 = 1 \Rightarrow a^2 - 1 = 0 \Rightarrow (a - 1)(a + 1) = 0 \Rightarrow a - 1 = 0\) or \(a + 1 = 0\) (since there are no divisors of zero) \(\Rightarrow a = 1\) or \(a = -1\).

(b) Suppose \(D\) has only finitely many units. 1 and \(-1\) are units, each of which is its own inverse. For any unit \(u \neq \pm 1\), we have \(u \neq u^{-1}\) from part (a). Hence the product of all the units of \(D\) is
\[1 \times (-1) \times (\text{a finite number of products } uu^{-1} \text{ with } u \neq u^{-1}),\]
which is clearly equal to \(-1\).

(c) This follows immediately from (b), since the units of \(\mathbb{Z}_n\) are the elements \(a \in \mathbb{Z}\) with \(1 \leq a < n\) and \((a, n) = 1\).

10. \(R\) is the principal ideal \(\langle 2 \rangle\) in \(\mathbb{Z}_{10}\), so \(R\) is a (commutative) subring of \(\mathbb{Z}_{10}\). The element \(1_R = 6\) is an identity for \(R\), and each non-zero element is a unit (you can check that \(2^{-1} = 8\), \(4^{-1} = 4\), \(8^{-1} = 2\) in \(R\)). Hence \(R\) has no divisors of zero, so \(R\) is a field.

11. (a) If \(E\) is a subfield of \(F\) then conditions (i), (ii) and (iii) hold, as they hold in any field.
   Conversely, suppose (i), (ii) and (iii) hold. Then \(E\) is a subring of \(F\) by conditions (i) and (ii). Since \(F\) has no divisors of zero, nor does \(E\); and since \(F\) is commutative, so is \(E\). As \(1 \in E\) it follows that \(E\) is an ID. By (iii) every non-zero element of \(E\) has a multiplicative inverse in \(E\), so \(E\) is a field.
(b) Clearly $E \subseteq \mathbb{R}$, and $E \neq \emptyset$ since $1 \in E$. Now let $\alpha, \beta \in E$, say $\alpha = a + b\sqrt{3}$ and $\beta = c + d\sqrt{3}$, where $a, b, c, d \in \mathbb{Q}$. Then $\alpha - \beta = (a - c) + (b - d)\sqrt{3} \in E$ and $\alpha \beta = (ac + 3bd) + (ad + bc)\sqrt{3} \in E$. Now suppose $\alpha \neq 0$. Then $(a, b) \neq (0, 0)$, and hence $a^2 - 3b^2 \neq 0$ (why?). Then

$$\alpha^{-1} = \frac{1}{a + b\sqrt{3}} = \frac{a - b\sqrt{3}}{a - b\sqrt{3}} \cdot \frac{1}{a + b\sqrt{3}} = \frac{a}{a^2 - 3b^2} + \frac{-b}{a^2 - 3b^2} \sqrt{3}$$

and hence $\alpha^{-1} \in E$. By the Subfield Test in (a), $E$ is a subfield of $\mathbb{R}$.

(c) Let $\alpha, \beta \in E$ be as in part (b). Then

$$\varphi(\alpha + \beta) = \varphi((a + c) + (b + d)\sqrt{3}) = ((a + c) + (b + d)X) + I = (a + bX + I) + (c + dX + I) = \varphi(\alpha) + \varphi(\beta),$$

so $\varphi$ preserves addition. Now

$$\varphi(\alpha \beta) = \varphi((ac + 3bd) + (ad + bc)\sqrt{3}) = (ac + 3bd) + (ad + bc)X + I.$$ 

Also

$$\varphi(\alpha)\varphi(\beta) = (a + bX + I)(c + dX + I) = ac + (ad + bc)X + bdX^2 + I = (ac + 3bd) + (ad + bc)X + bd(X^2 - 3) + I.$$ 

Since $X^2 - 3 \in I$, this is equal to $(ac + 3bd) + (ad + bc)X + I$. Hence $\varphi$ preserves multiplication.

Now let $a + b\sqrt{3} \in \ker \varphi$. Then $\varphi(a + b\sqrt{3}) = 0 \implies a + bX + I = I \implies a + bX \in I$. But every non-zero polynomial in $I$ has degree at least 2, so it follows that $a + bX$ must be the zero polynomial. Therefore $a = b = 0$ and hence $a + b\sqrt{3} = 0$. Thus $\ker \varphi = \{0\}$, and hence $\varphi$ is 1-1.

To show $\varphi$ maps $E$ onto $\mathbb{Q}[X]/I$, let $g(X) + I \in \mathbb{Q}[X]/I$. By the DA in $\mathbb{Q}[X]$, $\exists g(X), r(X) \in \mathbb{Q}[X]$ such that $g(X) = q(X)(X^2 - 3) + r(X)$ where $r(X) = 0$ or $\deg r(X) < 2$. In any case, we have $r(X) = a + bX$ for some $a, b \in \mathbb{Q}$. Then $g(X) + I = r(X) + I$ and hence $g(X) + I = \varphi(\alpha)$, where $\alpha = a + b\sqrt{3} \in E$.

It now follows that $\varphi$ is an isomorphism.
12. Since $|F| = 2^n$, then $2^n \cdot a = 0$ for all $a \in F, a \neq 0$, and hence $\text{char}(F)$ divides $2^n$. But $F$ is a field, so $\text{char}(F)$ must be zero or prime. The only possibility is $\text{char}(F) = 2$.

13. (1) False, e.g., take $R = \mathbb{Z}_{10}$ and $S = \{0, 2, 4, 6, 8\}$.
(2) False, e.g., take $R = \mathbb{Z}$ and $I = \langle 3 \rangle$.

14. (a) Let $a, b \in R, a \neq 0, b \neq 0$ and $n \in \mathbb{Z}^+$. Since $R$ is commutative, we have $(na)b = (nb)a$. If $na = 0$, then $(nb)a = 0$ and hence $nb = 0$ (since $R$ has no divisors of zero). Similarly, if $nb = 0$ then $na = 0$. Hence $na = 0 \iff nb = 0$, so it follows that $a, b$ have the same (additive) order.

(b) If $a \in D$, then $p \cdot a = 0$ in $D$, and hence $p \cdot (a + I) = 0$ in $D/I$, so every element of $D/I$ has additive order dividing $p$. Therefore every element of $D/I$ has order 1 or $p$, and hence $\text{char}(D/I) \leq p$. The only element of order 1 is the zero element $I$. Since $I \neq D$, the ring $D/I$ is not the zero ring, so must contain a non-zero element which must have order $p$. It now follows that $\text{char}(D/I) = p$.

If $\text{char}(D) = 0$, $D/I$ can have any characteristic. For example, if $D = \mathbb{Z}[X]$ and $I = \langle X \rangle$, then $D/I \cong \mathbb{Z}$ has characteristic = 0; but if $p$ is prime, $D = \mathbb{Z}$ and $I = \langle p \rangle$, then $D/I \cong \mathbb{Z}_p$ has characteristic = $p$.

15. (a) If $a, b \in R$, the Binomial Theorem gives

$$(a + b)^p = a^p + \sum_{i=1}^{p-1} \binom{p}{i} a^i b^{p-i} + b^p.$$ 

As $p$ divides $\binom{p}{i}$ for all $i \in \{1, 2, \ldots, p - 1\}$, and $\text{char}(R) = p$ it follows that $\binom{p}{i} a^i b^{p-i} = 0$ for $i = 1, 2, \ldots, p - 1$. Hence

$$(a + b)^p = a^p + 0 + b^p = a^p + b^p.$$ 

This shows that $\varphi(a + b) = \varphi(a) + \varphi(b)$. Also $\varphi(ab) = (ab)^p = a^p b^p = \varphi(a) \varphi(b)$ since $R$ is commutative. Hence $\varphi$ is a homomorphism.

(b) Let $R$ be a finite field of characteristic $p$. Then $a^p = 0 \implies a = 0$ since $R$ has no divisors of zero. Hence $\ker \varphi = \{0\}$ and so $\varphi$ is 1-1. Since $R$ is finite, $\varphi$ also maps $R$ onto $R$, and hence $\varphi$ is an automorphism.
16. We show that the field of quotients of \( D \) is \( \mathbb{Q} + \mathbb{Q}i = \{ x + yi \mid x, y \in \mathbb{Q} \} \).

Let \( n + mi, r + si \in D \). Then

\[
\frac{n + mi}{r + si} = \frac{n + mi}{r + si} \cdot \frac{r - si}{r - si} = \frac{(nr + ms) + (mr - ns)i}{r^2 + s^2} = \frac{nr + ms}{r^2 + s^2} + \frac{mr - ns}{r^2 + s^2}i.
\]

Since \( \frac{nr + ms}{r^2 + s^2} \) and \( \frac{mr - ns}{r^2 + s^2} \) are in \( \mathbb{Q} \), this shows that every quotient of elements of \( D \) belongs to \( \mathbb{Q} + \mathbb{Q}i \). On the other hand, if \( x + yi \in \mathbb{Q} + \mathbb{Q}i \), say \( x = a/b, y = c/d \) where \( a, b, c, d \in \mathbb{Z} \), then

\[
x + yi = \frac{a}{b} + \frac{c}{d}i = \frac{ad + (bc)i}{bd},
\]

which is a quotient of two elements of \( D \). The result now follows.

17. (a) \( D \) is clearly closed under subtraction and is non-empty. If \( x = a + b\alpha + c\alpha^2 \) and \( y = d + e\alpha + f\alpha^2 \) belong to \( D \) (where \( a, b, c, d, e, f \in \mathbb{Z} \)), then (using \( \alpha^3 = 2 \))

\[
xy = ad + (bd + ae)\alpha + (cd + be + af)\alpha^2 + (bf + ce)\alpha^3 + (cf)\alpha^4
\]

\[
= ad + (bd + ae)\alpha + (cd + be + af)\alpha^2 + (bf + ce)2 + (cf)2\alpha + (cd + be + af)\alpha^2
\]

belongs to \( D \), so \( D \) is closed under multiplication and is therefore a subring of \( \mathbb{R} \). Since \( \mathbb{R} \) is commutative and has no divisors of zero, the same is true for \( D \). Also \( 1 = 1 + 0\alpha + 0\alpha^2 \in D \), so \( D \) is an integral domain.

(b) Suppose \( u = a + b\alpha + c\alpha^2 \in D \), \( u \neq 0 \). The only hard part here is to show that \( u^{-1} \in F \) (the rest of the proof goes much the same as in question 18 above). Later we will develop some theory which will make this easy, but for now we adopt a bare hands approach. We first prove a couple of lemmas which will make the final proof go more smoothly.

**Lemma 1.** If \( a, b, c \in \mathbb{Q} \), then \( a + b\alpha + c\alpha^2 = 0 \) \( \iff \) \( a = b = c = 0 \).

**Proof.** Suppose \( \exists a, b, c \in \mathbb{Q} \), not all zero, such that \( a + b\alpha + c\alpha^2 = 0 \). Then \( g(X) = a + bX + cX^2 \in \mathbb{Q}[X] \) is a non-zero polynomial of degree \( \leq 2 \) which has \( \alpha \) as a zero. We will show this is impossible.
Clearly \( \alpha \) is a zero of the polynomial \( f(X) = X^3 - 2 \in \mathbb{Z}[X] \). By the Rational Zeros Theorem (see question 25 below), the only possible rational zeros of \( f(X) \) are \( \pm 1, \pm 2 \). But none of these is a zero of \( f(X) \), so \( f(X) \) has no rational zeros. Hence \( \alpha \) must be irrational.

If \( b = c = 0 \), then \( a \neq 0 \) and clearly \( \alpha \) is not a zero of \( g(X) = a \).

If \( \deg g(x) = 1 \), then \( g(X) = a + bX \) where \( b \neq 0 \). If \( g(\alpha) = 0 \), then \( a + b\alpha = 0 \implies \alpha = -a/b \in \mathbb{Q} \), contradiction.

So we can assume \( \deg g(x) = 2 \). Then the DA in \( \mathbb{Q}[X] \) gives

\[
(*) \quad X^3 - 2 = g(X)q(X) + r(X)
\]

for some \( q(X), r(X) \in \mathbb{Q}[X] \) with \( r(X) = 0 \) or \( \deg r(X) \leq 1 \). If \( r(X) \neq 0 \), evaluating (*) at \( X = \alpha \) gives \( 0 = 0.q(\alpha) + r(\alpha) \), so \( r(\alpha) = 0 \), contradicting what has already been proved. Hence we must have \( r(X) = 0 \) so that \( X^3 - 2 = g(X)q(X) \). But \( g(X) \in \mathbb{Q}[X] \) has degree 1 and therefore has exactly one rational zero \( s \).

But then \( s \) is a zero of \( X^3 - 2 \), contradicting what has already been proved.

This proves the \( \implies \) direction. The other direction is obvious.

**Lemma 2.** If \( a, b, c \in \mathbb{Z} \) and \( a^3 + 2b^3 + 4c^3 - 6abc = 0 \), then \( a = b = c = 0 \).

**Proof.** Suppose \( \exists a, b, c \in \mathbb{Z} \) which are not all zero and satisfy

\[
(**) \quad a^3 + 2b^3 + 4c^3 - 6abc = 0.
\]

If \( \gcd(a, b, c) = g > 1 \), we can write \( a = ga', g = gb', c = gc' \) where \( \gcd(a', b', c') = 1 \). Substituting into (***) and cancelling out the common factor \( g^3 \), we see that \( a', b', c' \in \mathbb{Z} \) also give a non-trivial solution of (**). This shows that we can assume (without loss of generality) that our solution \((a, b, c)\) satisfies \( \gcd(a, b, c) = 1 \).

Reducing (***) \( \mod 2 \) shows that \( a^3 \equiv 0 \pmod 2 \), so \( a \) is even, and hence \( a = 2a_1 \) for some \( a_1 \in \mathbb{Z} \). Substituting this into (***) and cancelling out the common factor 2, we obtain \( 4a_1^3 + b^3 + 2c^3 - 6a_1bc = 0 \). The same reasoning then shows that \( b \) must be even,
say \( b = 2b_1 \). Substituting into (**) and cancelling by 2, we obtain
\[
2a_1^3 + 4b_1^3 + c^3 - 6a_1b_1c = 0. 
\]
Then (as above) we see that \( c \) is even. But then \( a, b, c \) are all divisible by 2, contradicting our assumption that \( \gcd(a, b, c) = 1 \). The result now follows. \( \Box \)

Now let \( u = a + b\alpha + c\alpha^2 \in D, u \neq 0 \). Then \( a, b, c \in \mathbb{Z} \) and are not all zero (by Lemma 1). If \( x, y, z \in \mathbb{Q} \), then using \( \alpha^3 = 2 \) we find
\[
(a + b\alpha + c\alpha^2)(x + y\alpha + z\alpha^2) = (ax + 2cy + 2bz) + (bx + ay + 2cz)\alpha + (cx + by + az)\alpha^2. 
\]
By Lemma 1 we then have
\[
(a + b\alpha + c\alpha^2)(x + y\alpha + z\alpha^2) = 1 \iff \begin{cases} \quad ax + 2cy + 2bz = 1 \\ \quad bx + ay + 2cz = 0 \\ \quad cx + by + az = 0 \end{cases} 
\]
This is a \( 3 \times 3 \) system of linear equations in the unknowns \( x, y, z \)
with coefficient matrix
\[
A = \begin{bmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix} .
\]
The determinant of \( A \) is \( \det(A) = a^3 + 2b^3 + 4c^3 - 6abc \), which is \( \neq 0 \) by Lemma 2. It follows that there is a unique solution for \( x, y, z \).
Also \( x, y, z \in \mathbb{Q} \) since all the coefficients and RHS constants in the system of linear equations are integers. So this proves (finally!) that \( u^{-1} = x + y\alpha + z\alpha^2 \in F \).

18. (a) Let \( f \) be an automorphism of \( \mathbb{Q} \). Then \( f(1) \neq 0 \), or else \( f(x) = f(x \cdot 1) = f(x)f(1) = 0 \) for all \( x \in \mathbb{Q} \), so \( f \) would not map \( \mathbb{Q} \)
onto itself. Also \( f(1) = f(1 \cdot 1) = f(1)f(1) \), so \( f(1) = 1 \) by the
cancellation law in \( \mathbb{Q} \). If \( n \in \mathbb{Z}^+ \), then (since \( f \) preserves addition)
\( f(n) = f(n \cdot 1) = nf(1) = n \), and \( f(-n) = -f(n) = -n \). Since
also \( f(0) = 0 \), it follows that \( f(n) = n \) for all \( n \in \mathbb{Z} \). Now let
\( x = m/n \in \mathbb{Q} \), where \( m, n \in \mathbb{Z} \) and \( n > 0 \). Then
\[
m = f(m) = f(n \cdot \frac{m}{n}) = nf(\frac{m}{n}),
\]
so \( f(\frac{m}{n}) = \frac{m}{n} \). In other words, \( f \) is the identity map on \( \mathbb{Q} \).
(b) Now let $f$ be an automorphism of $\mathbb{R}$. The same reasoning as in (a) shows that $f(x) = x$ for all $x \in \mathbb{Q}$.

We now show that the map $f$ is order preserving. Suppose $a, b \in \mathbb{R}$ and $a < b$. Then $b - a > 0$ so $b - a = r^2$ for some $r \in \mathbb{R}$. Then

$$f(b) - f(a) = f(b - a) = f(r^2) = f(r)^2 \geq 0.$$  

We cannot have $f(r) = 0$, or else $f(b) = f(a)$, contradicting the fact that $f$ is 1-1. Hence $f(r) \neq 0$, so $f(b) - f(a) > 0$ and hence $f(a) < f(b)$, as required.

Finally, suppose $x \in \mathbb{R}$ and $f(x) \neq x$, say $x < f(x)$. Then $\exists r \in \mathbb{Q}$ such that $x < r < f(x)$. But then $x < r \implies f(x) < f(r) = r$, contradiction. Hence we must have $f(x) = x$ for all $x \in \mathbb{R}$ as claimed.

19. (1) $q(X) = X^4 + X^3 + X^2 + X - 2$, $r(X) = 4X + 3$.
   (2) $q(X) = 6X^4 + 7X^3 + 2X^2 - X - 2$, $r(X) = 4$.

   (2) $(X - 3)^2(X + 1)$.

21. Let $F$ be a field and $I \triangleleft F[X]$. If $I = \{0\}$, then $I = \langle 0 \rangle$.

Now suppose $I \neq \{0\}$ and let $d(X)$ be a polynomial of least degree in $I$. Then $\langle d(X) \rangle \subseteq I$. If $b(X) \in I$, then by the Division Algorithm in $F[X]$, there exist $q(X), r(X) \in F[X]$ such that $b(X) = q(X)d(X) + r(X)$ and $r(X) = 0$ or $\deg r(X) < \deg d(X)$. If $r(X) \neq 0$, then $r(X) = b(X) - q(X)d(X) \in I$ and has degree $< \deg d(X)$, contradicting the definition of $d(X)$. Hence we must have $r(X) = 0$, so that $b(X) = q(X)d(X) \in \langle d(X) \rangle$. It now follows that $I = \langle d(X) \rangle$.

22. (a) No. For example, take $f(X) = X$ and $g(X) = 2X$. If $\exists q(X), r(X) \in \mathbb{Z}[X]$ such that

$$(*) \quad X = 2X \cdot q(X) + r(X) \text{ where } r(X) = 0 \text{ or } \deg r(X) < \deg 2X,$$

then $r(X) = a_0$ for some $a_0 \in \mathbb{Z}$. Evaluating $(*)$ at $X = 0$ gives $r(X) = a_0 = 0$. Then evaluating at $X = 1$ gives $1 = 2q(1)$, which is impossible since $q(1) \in \mathbb{Z}$.
(b) No. For example, take \( f(X, Y) = X \) and \( g(X, Y) = Y \). If 
\[ \exists q(X, Y), r(X, Y) \in \mathbb{Z}[X, Y] \] such that 
\[ (\star) \quad X = Yq(X, Y) + r(X, Y) \] where \( r(X, Y) = 0 \) or \( \deg r(X, Y) < \deg Y \), 
then \( r(X, Y) = a_0 \) for some \( a_0 \in \mathbb{Z} \). Evaluating \( (\star) \) at \( Y = 0 \) gives 
\( X = a_0 \), which is impossible.

23. Suppose \( r, s \in \mathbb{Z} \), \( (r, s) = 1 \) and \( f(r/s) = 0 \). Then 
\[ a_n(r/s)^n + a_{n-1}(r/s)^{n-1} + \cdots + a_1(r/s) + a_0 = 0. \]

Multiplying both sides by \( s^n \) gives 
\[ a_nr^n + a_{n-1}r^{n-1}s + \cdots + a_1rs^{n-1} + a_0s^n = 0 \quad \text{in } \mathbb{Z}, \]
and hence \( s \mid a_nr^n \). Since \( (r, s) = 1 \), it follows that \( s \mid a_n \). Similarly, 
\( r \mid a_0 \).