1. An extrusion process produces steel pipes of length ten metres, which are then cut into five-two metre pieces. The thickness of the steel pipes is a critical quality measure. As part of the quality control, range and x-bar charts are to be produced, using as subgroup base all five pieces of pipes from a single extrusion.

(a) Briefly discuss this sampling scheme and its impact on the ability of the charts to identify assignable causes of variation.

(b) Suggest a better sampling scheme.

Solution:

(a) The sub-samples are not independent here. This leads to three problems.

i. Measurements within each groups will be less variable. Thus the true variability of the process will not be reflected. Thus the range charts may deem the process in control because of the low variability, although the process may be out of control.

ii. The standard deviation $\sigma$ will be under-estimated.

iii. Since the data is correlated, the variance of the sample means is no longer found by $\sigma^2/m$. There is no way to compute this variance since the correlation structure is unknown.

(b) A better sampling scheme is to take a random sample with one observation from each set of parts produced by the machine.

2. At regular intervals, samples of size $n = 8$ are taken from the output of a process. A quality characteristic, which can be assumed Normally distributed, is measured. For each of 50 samples, the sample mean and sample range is determined with the results yielding

$$\sum_{i=1}^{50} \bar{x}_i = 2000 \quad \text{and} \quad \sum_{i=1}^{50} R_i = 250.$$  

[Hint: You will need to use the values $d_2(8) = 2.847$ and $d_3(8) = 0.820$.)

(a) Calculate values for the commonly used form of control limits for the $\bar{x}$ chart and for the $R$ chart based on the above data.

Solution:

$$\overline{x} = \frac{2000}{50} = 40, \quad \overline{R} = \frac{250}{50} = 5.$$  

$$\hat{\sigma} = \frac{\overline{R}}{d_2(8)} = \frac{5}{2.847} = 1.756.$$  

$\bar{x}$ chart

Centreline is at $\overline{x} = 40$

$$UCL = \overline{x} + 3 \frac{\hat{\sigma}}{\sqrt{m}}$$

$$= 40 + 3 \times \frac{5}{2.847} \times \frac{1}{\sqrt{8}}$$

$$= 41.86.$$  

$$LCL = \overline{x} - 3 \frac{\hat{\sigma}}{\sqrt{m}}$$

$$= 40 - 3 \times \frac{5}{2.847} \times \frac{1}{\sqrt{8}}$$

$$= 38.14.$$  

1
R chart

centreline is at $r = 5$.

$$UCL = r + 3 \times d_3(8) \times \hat{\sigma}$$
$$= 5 + 3(0.820) \times \frac{5}{2.847}$$
$$= 9.32.$$  

$$LCL = r - 3 \times d_3(8) \times \hat{\sigma}$$
$$= 5 - 3(0.820) \times \frac{5}{2.847}$$
$$= 0.680.$$  

(b) Assume that the charts constructed using these limits suggest the process is in control. Estimate the natural tolerance limits for this process.

Solution:

$$\hat{UNTL} = \hat{\mu} + 3 \hat{\sigma}$$
$$= \bar{r} + 3 \bar{r} \frac{r}{c_2(8)}$$
$$= 40 + \frac{(3)(5)}{2.847}$$
$$= 45.27.$$  

$$\hat{LNTL} = 40 - \frac{(3)(5)}{2.847}$$
$$= 34.73.$$  

(c) If the specification limits are $41.0 \pm 5.0$ what can you say about the ability of the process to meet these limits?

Solution:

Assume that the process is centred. Then

$$c_p = \frac{USL - LSL}{6\sigma} = \frac{10}{6 \times 1.756} = 0.949 < 1,$$

so the process is not able to meet these limits.

Since the process mean is $\mu = 40$ while the specified mean is 41, the process is not centred. We have $USL = 46$, $LSL = 36$, so

$$c_{pu} = \frac{USL - \mu}{3\sigma} = \frac{46 - 40}{3 \times 1.756} = 1.139$$

$$c_{pf} = \left| \frac{\mu - LSL}{3\sigma} \right| = \frac{40 - 36}{3 \times 1.756} = 0.759$$

Then $c_{pk} = \min(c_{pu}, c_{pf}) = 0.759$, and it is clear that the process is not able to meet the specification limits.
(d) Assume that an item whose quality characteristic is above the USL can be reworked, whilst one below the LSL must be scrapped. Estimate the percentage of items produced by this process in each category.

Solution:

\[
P(X > 46) \approx P \left( \frac{Z > \frac{46 - 40}{\sqrt{5/2.847}}}{\sqrt{5/2.847}} \right) = P(Z > 4.53) = 0.00000298.
\]

\[
P(X < 36) \approx P \left( \frac{Z < \frac{36 - 40}{\sqrt{5/2.847}}}{\sqrt{5/2.847}} \right) = P(Z < -3.018) = 0.0013.
\]

(e) Make some suggestions as to how the performance of this process could be improved.

Solution:

The process is not centred about the required mean of 40. Thus shifting down the mean of the process will reduce the proportion of non-conforming items produced. However, this will still not be satisfactory, as the the natural tolerance limits will still lie outside the specification limits. The only way to improve the performance of the process is to reduce the variability, that is, reduce \( \sigma \).

3. The following table gives the values of \( \bar{X} \) and \( R \) for 24 samples of size \( n = 5 \) taken from a process that produces bearings. The measurements are of the inside diameter of each bearing, with only the last three digits recorded: thus, for example, 34.5 represents 0.50345.

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>( \bar{X} )</th>
<th>( R )</th>
<th>Sample No.</th>
<th>( \bar{X} )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>34.5</td>
<td>3</td>
<td>13</td>
<td>35.4</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>34.2</td>
<td>4</td>
<td>14</td>
<td>34.0</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>31.6</td>
<td>4</td>
<td>15</td>
<td>37.1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>31.5</td>
<td>4</td>
<td>16</td>
<td>34.9</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>35.0</td>
<td>5</td>
<td>17</td>
<td>33.5</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>34.1</td>
<td>6</td>
<td>18</td>
<td>31.7</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>32.6</td>
<td>4</td>
<td>19</td>
<td>34.0</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>33.8</td>
<td>3</td>
<td>20</td>
<td>35.1</td>
<td>4</td>
</tr>
<tr>
<td>9</td>
<td>34.8</td>
<td>7</td>
<td>21</td>
<td>33.7</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>33.6</td>
<td>8</td>
<td>22</td>
<td>32.8</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>31.9</td>
<td>3</td>
<td>23</td>
<td>33.5</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>38.6</td>
<td>9</td>
<td>24</td>
<td>34.2</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Construct an \( R \) chart and an \( \bar{X} \) chart for this process. You should do the calculations either by hand or using Excel.

Solution:

Calculations give

\[
\bar{r} = 4.708, \quad \hat{\sigma} = \frac{\bar{r}}{d_2(n)} = \frac{4.608}{2.326} = 2.024.
\]

**X-bar chart**

centre line is at \( \bar{X} = 34.0 \)

\[
\text{LCL} = \bar{X} - 3 \frac{\hat{\sigma}}{\sqrt{5}} = 34.0 - 3(2.024)\sqrt{5} = 31.3
\]
\[ UCL = \bar{x} + 3 \frac{\hat{\sigma}}{\sqrt{5}} = 34.0 + 3(2.024)\sqrt{5} = 36.7 \]

**Range chart**

centre line is at \( \bar{r} = 4.7 \)

\( \hat{\sigma}_R = d_3(5) \hat{\sigma} = 0.864 \times 2.024 \)

\[ LCL = \bar{r} - 3 \hat{\sigma}_R = 4.708 - (3)(0.864)(2.024) = -0.54, \]

so the LCL is at 0.

\[ UCL = \bar{r} + 3 \hat{\sigma}_R = 4.708 + (3)(0.864)(2.024) = 9.96 \]

The control charts are plotted below.

(b) Does the process appear to be in statistical control? Comment briefly, and indicate what it would be appropriate to do next. (There is no need to do more.)

Solution:
The range chart appears to be in control, so the overall variability in the process seems to be in control. However, note that sample 12 has the highest range $R = 9$, and the samples at both ends of the chart have smaller ranges. In particular, it seems that the ranges drift downward in the last few samples.

The X-bar chart shows that samples 12 and 15 are beyond the control limits, so the process is out of control. Special cause effects should be sought for these samples, especially sample 12 as it also showed a large range on the R chart.

Once the special cause effects have been identified for samples 12 and 15, these samples can be deleted and the control limits re-calculated for subsequent use on both charts.

(c) Indicate how to determine control limits for the $R$ chart by the direct probability method. Explain your method by reference to appropriate probabilities for the range $Z_{(n)} - Z_{(1)}$ based on $n$ independent and identically distributed standard Normal random variables. For comparison add the resultant limits to your earlier chart.

Solution:

Now we need to find $z_u$ and $z_l$ such that

$$P\left(Z_{(5)} - Z_{(1)} > z_u\right) = P\left(Z_{(5)} - Z_{(1)} < z_l\right) = 0.001.$$  

The values of $z_u$ and $z_l$ can be found by order statistics, or simulations. By simulations (using SPSS we simulate 5 columns of 1,000,000 values from a $N(0, 1)$ distribution and treat each row as a sample of size 5. We find the row range, sort this, and find the 1000th largest and smallest values. These are the required upper and lower limits.) we find $z_u = 5.49$ and $z_l = 0.36$. Then

$$UCL = z_u \hat{\sigma} = 5.49 \times 2.024 = 11.1$$

$$LCL = z_l \hat{\sigma} = 0.36 \times 2.024 = 0.73.$$  

These are plotted on the original chart for comparison.

(d) If the specifications on this diameter are 0.5030 ± 0.0010, and the process output is Normally distributed, estimate the percentage of non-conforming bearings produced.

Solution:

Let the random variable $X$ denote the “inside diameter” measurement of a typical bearing. Then the proportion of non-conforming bearings produced is given by $P(X > USL) + P(X < LSL)$. Note that the data records the last three digits of the measurements, so on this scale the specifications are $30 \pm 10$,  

5
that is, the USL is 40 and the LSL is 20. Since the measurements are normally distributed with mean and standard deviation as estimated from the data, we have

\[ P(X > USL) + P(X < LSL) \approx P(Z > \frac{40 - 34}{2.024}) + P(Z > \frac{20 - 34}{2.024}) \]

\[ = P(Z > 2.964) + P(Z < -6.917) \]

\[ = 0.0015 + 0.000 \]

\[ = 0.0015, \]

so the percentage of non-conforming bearing produced is 0.15%.

©Nazim Khan, August 2009