1. Find the three sigma control limits for a
   
   (a) a $c$ chart with process average equal to four defectives.

   **Solution:**

   A $c$ chart plots the number of defectives, where the data $X_1, X_2, \ldots, X_n$ are modelled as independent $\text{Poi}(\lambda)$ random variable. Then

   \[
   E(X_i) = \text{Var}(X_i) = \lambda, \quad i = 1, 2, \ldots, n,
   \]

   and the control lines for the chart are:

   - **centreline:** $\bar{\lambda} = 4$
   - **UCL:** $\lambda + 3\sqrt{\lambda} = 4 + 6 = 10$
   - **LCL:** $\lambda - 3\sqrt{\lambda} = 4 - 6 = -2$

   so $\text{LCL} = 0$.

   where $\bar{\lambda} = \bar{x}$, the mean number of defectives in the sample.

   (b) a $u$ chart with $\bar{c} = 4$ and $n = 4$.

   **Solution:**

   A $u$ chart plots the number of defectives per unit. Here the data are $X_1, X_2, \ldots, X_n$ where $X_i \sim \text{Poi}(\lambda)$ are the number of defectives in a sample of size $m$. Put $U_i = X_i / m$. Then

   \[
   E(U_i) = \lambda / m \approx \bar{\lambda} / m = \bar{c}, \quad \text{Var}(U_i) = \lambda / m^2 \approx \bar{\lambda} / m^2 = \bar{c} / m.
   \]

   The the control lines are:

   - **Centreline:** $\bar{\pi}$
   - **UCL:** $\bar{\pi} + 3\sqrt{\bar{\pi}/m}$
   - **LCL:** $\bar{\pi} - 3\sqrt{\bar{\pi}/m}$
2. Control charts for $\bar{x}$ and $R$ are to be established to control the tensile strength of a metal part. Assume that the tensile strength is normally distributed. Thirty samples of size $m = 6$ parts are collected over a period of time with the following results:

$$\sum_{i=1}^{30} x_i = 6,000 \sum_{i=1}^{30} R_i = 150.$$ 

(a) Calculate control limits for $\bar{x}$ and $R$.

Solution:

$\bar{x} = \frac{6000}{30} = 200, \tau = \frac{150}{30} = 5, \text{ and } \hat{\sigma} = \frac{\tau}{d_2(6)} = \frac{5}{2.534} = 1.973.$ The control lines for the $\bar{X}$ chart are:

Centreline : $\bar{x} = 200$

UCL : $\bar{x} + 3\hat{\sigma}/\sqrt{6} = 200 + \frac{3\times5}{2.534 \times \sqrt{6}} = 202.4$

LCL : $\bar{x} - 3\hat{\sigma}/\sqrt{6} = 200 - \frac{3\times5}{2.534 \times \sqrt{6}} = 197.6$

For the R chart, the control lines are:

Centreline : $\bar{r} = 5$

UCL : $\bar{r} + 3\hat{\sigma}_R = \bar{r} + 3d_3(6)\hat{\sigma} = 5 + 3(0.848) \frac{5}{2.534} = 10.0$

LCL : $\bar{r} - 3\hat{\sigma}_R = \bar{r} - 3d_3(6)\hat{\sigma} = 5 - 3(0.848) \frac{5}{2.534} = -0.02$

so LCL is at 0.

(b) Both charts exhibit control. The specifications on tensile strength are $200 \pm 5$. What are the conclusions regarding process capability?

Solution:

$$C_p = \frac{USL - LSL}{6\hat{\sigma}}, \quad C_p = \frac{10}{6(5/2.534)} = 0.8447.$$ 

The process is not capable of meeting the specifications — the specification range is less than $6\hat{\sigma}$.

(c) What are the natural tolerance limits of the process? Does the process meet the specifications? Use a suitable diagram to support your conclusion.
Solution:

\[ \text{UNTL} = \bar{x} + 3\sigma = 200 + \frac{3 \times 5}{2.534} = 205.9 \]
\[ \text{LNTL} = \bar{x} - 3\sigma = 200 - \frac{3 \times 5}{2.534} = 194.1 \]

This information is presented in the diagram below.

<table>
<thead>
<tr>
<th>LNTL</th>
<th>LSL</th>
<th>( \bar{x} )</th>
<th>USL</th>
<th>UNTL</th>
</tr>
</thead>
<tbody>
<tr>
<td>194.1</td>
<td>195</td>
<td>200</td>
<td>205</td>
<td>205.9</td>
</tr>
</tbody>
</table>

(d) What proportion of non-conforming items are produced by the process?

Solution:

Proportion non-conforming = \( P(X > 205) + P(X < 195) \)

where \( X \sim N(\mu, \sigma^2) \)

\[ \approx P\left( Z > \frac{205 - 200}{\sigma} \right) + P\left( Z < \frac{195 - 200}{\sigma} \right) \]
\[ = P(Z > 2.534) + P(Z < -2.534) \]
\[ = 2 \times 0.0056 \]
\[ = 0.112. \]

(e) Find the average run length for the process.

Solution:

The probability that a point on an \( \bar{X} \) chart or an R chart lies outside the control limits is 0.0027. Then the average run length is

\[ E(R) = \frac{1}{\gamma} = \frac{1}{0.0027} = 370. \]

3. The fraction of defectives for a given process is 0.1. A \( p \) chart is to be constructed based on inspection of 20 items from each successive batch produced.

(a) What would be the control limits for such a Shewhart chart?
Solution:

Centre line \( p = 0.1 \)

\[
UCL = p + 3\sqrt{\frac{p(1-p)}{n}} = 0.1 + \sqrt{\frac{0.1 \times 0.9}{20}} = 0.301
\]

\[
LCL = p - 3\sqrt{\frac{p(1-p)}{n}} = 0.1 - \sqrt{\frac{0.1 \times 0.9}{20}} = -0.101,
\]

so \( LCL = 0 \).

(b) What is the probability that the process is deemed “out of control” through the fraction of defectives for the next batch being strictly outside the control limits?

Solution:

For the process to be deemed out of control we need \( \hat{p} = X/20 > 0.301 \) where \( X \sim \text{Bin}(20, 0.1) \), that is, the point should lie above the UCL. Thus we need \( X > 20 \times 0.301 = 6.02 \geq 7 \), that is, \( P(X \geq 7) = 0.002386. \)

(c) How often on average would a “false alarm” be raised for the process?

Solution:

The ARL is \( E(R) = 1/\gamma = 1/0.002386 = 419.1 \), that is every 420 samples.

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