Test 4 — Mock Mid-Semester Test

This test counts for 15% of your final mark for MATH1050 Calculus C.

There are FOUR questions in this test, worth a total of 50 marks.

Attempt all questions.

Marks are given for clarity and correctness of method, not just for correct answers.

Appropriate working with reasons should be given.

Write your answers in the blank space following the question.

The last page has been left blank for rough working.

Time Limit: 50 minutes
1. Linear Equations (11 marks)

(a) Solve the following equation for $x$.

\[
\frac{x - 3}{3} + 1 = \frac{x + 3}{6}
\] 

(3 marks)

\[
(x - 6) \iff 6 \left( \frac{x - 3}{3} + 1 \right) = 6 \cdot \frac{x + 3}{6}
\]

\[
\iff 6 \cdot \frac{x - 3}{3} + 6 = \frac{x + 3}{6}
\]

\[
\iff 2 (x - 3) + 6 = x + 3
\]

\[
\iff 2x - 6 + 6 = x + 3
\]

\[
\iff 2x = x + 3
\]

\[
\iff x = 3
\]
(b) How many litres of a 80% hydrochloric acid solution should be added to a 40% hydrochloric acid solution to make 4 litres of a 50% solution? (4 marks)

<table>
<thead>
<tr>
<th>Vol</th>
<th>80%</th>
<th>40%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td>$A - \chi$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Vol of HCl</td>
<td>0.8$\chi$</td>
<td>0.4$(A - \chi)$</td>
<td>0.5(4)</td>
</tr>
</tbody>
</table>

\[0.8\chi + 0.4(A - \chi) = 0.5(4)\]

\[0.8\chi + 1.6 - 0.4\chi = 2\]

\[0.4\chi = 2 - 1.6\]

\[0.4\chi = 0.4\chi\]

\[\chi = \frac{0.4}{0.4} = 1\]

1 L of 80%

3 L of 40%
A landscaping company placed two orders with a nursery. The first order was for thirteen bushes and four trees costing a total of $487. The second order was for six bushes and two trees costing a total of $232. The bill did not list the cost of each item. What was the cost of each bush and tree?

Let $b =$ cost of a bush

Let $t =$ cost of a tree

\[
\begin{align*}
13b + 4t &= 487 \quad (1) \\
6b + 2t &= 232 \quad (2)
\end{align*}
\]

\[
\Leftrightarrow \begin{align*}
13b + 4t &= 487 \quad (1) \\
12b + 4t &= 464 \quad (2) \Leftrightarrow (1) - (2)
\end{align*}
\]

From the original equation (2),

\[
6(23) + 2t = 232
\]

\[
\Leftrightarrow 138 + 2t = 232
\]

\[
\Leftrightarrow 2t = 94
\]

So $t = 47$

bushes cost $23 each, trees cost $47 each.
2. Factorisations and Quadratics (11 marks)

(a) Simplify

\[
\frac{6x^2 + 7x + 2}{4x^2 - 1} \div \frac{9x^2 + 18x + 8}{9x^2 - 16} \times \frac{4x^3 - x}{3x^2 - 4x}
\]

(5 marks)

\[
= \frac{6x^2 + 7x + 2}{4x^2 - 1} \cdot \frac{9x^2 - 16}{9x^2 + 18x + 8} \cdot \frac{4x^3 - x}{3x^2 - 4x}
\]

\[
= \frac{6x^2 + 7x + 2}{4x^2 - 1} \cdot \frac{9x^2 - 16}{9x^2 + 18x + 8} \cdot \frac{4x^3 - x}{3x^2 - 4x}
\]

\[
= \frac{(2x+1)(3x+2)}{(2x+1)(2x-1)} \cdot \frac{(3x+4)(3x-4)}{(3x+4)(3x+2)} \cdot \frac{(2x+1)(2x-1)}{(3x-4)(3x+2)}
\]

\[
= \frac{2x+1}{3x+2}
\]

9\(x^2 + 18x + 8\) : 

\[
= (3x+4)(3x+2)
\]

6\(x^2 + 7x + 2\) : 

\[
= (2x+1)(3x+2)
\]

\[
= (3x+4)(3x+2)
\]

\[
= (2x+1)(3x+2)
\]

\[
= 3(2x+1)(\frac{x+\frac{2}{3}}{3})
\]

\[
= 3(2x+1)(\frac{x+\frac{2}{3}}{3})
\]

\[
= (3x+4)(3x+2)
\]

\[
= (2x+1)(3x+2)
\]
(b) For what value(s) of \( k \) does the equation \( 36x^2 - 6kx + 1 = 0 \) have exactly one solution?

We need
\[
b^2 - 4ac = (-6k)^2 - 4(36)(1) = 0
\]
\[
\iff 36k^2 - 4(36) = 0 \]
\[
\iff k^2 - 4 = 0 \]
\[
\iff (k + 2)(-k - 2) = 0
\]
\[
\iff k = 2, -2
\]

(2 marks)

(c) A missile is fired up in the air at the edge of a 1000 m high cliff so the height \( h(t) \) of the missile above the cliff at time \( t \) (in seconds) is given by

\[
h(t) = 50t - 5t^2, \quad t \geq 0.
\]

The missile travels up in the air and then falls on the beach below. How long does the missile take to land on the beach below? (4 marks)

We need \( t \) when

\[
h(t) = -1000
\]
\[
-5t^2 + 50t = -1000
\]
\[
\iff 5t^2 - 50t + 1000 = 0
\]
\[
\iff t^2 - 10t - 200 = 0
\]
\[
\iff (t - 20)(t + 10) = 0
\]
\[
\iff t = 20 \text{ or } -10.
\]

Since \( t \geq 0 \), we have \( t = 20 \text{ s} \).
3. Exponents and Logarithms (16 marks)

(a) Simplify the following expressing with positive indices:

\[
\left( \frac{9x^2}{y^{-4}} \right)^{\frac{1}{2}} \div \sqrt[4]{\frac{16x^8}{81y^{-4}}} 
\]

\[
= \left( \frac{9x^2}{y^{-4}} \right)^{\frac{1}{2}} \cdot \left( \frac{16x^8}{81y^{-4}} \right)^{\frac{1}{4}} 
\]

\[
= \left( 3x^2 y^4 \right)^{\frac{1}{2}} \cdot \left( 81 y^{-4} \right)^{-\frac{1}{4}} 
\]

\[
= \frac{3x y^2}{2x^2} \cdot \frac{3y^{-1}}{2} 
\]

\[
= \frac{9y}{2x^2} 
\]
(b) Let \( x = \log_3 4 \) and \( y = \log_3 5 \). Express the following in terms of \( x \) and \( y \).

i. \( \log_3 20 \) 
\[
\log_3 20 = \log_3 (4 \cdot 5) \\
= \log_3 4 + \log_3 5 \\
= x + y
\]

ii. \( \log_3 0.75 \) 
\[
= \log_3 \frac{3}{4} \\
= \log_3 3 - \log_3 4 \\
= 1 - x
\]

iii. \( \log_3 48 \) 
\[
= \log_3 (16 \cdot 3) \\
= \log_3 16 + \log_3 3 \\
= \log_3 4^2 + 1
\]
(c) Solve the following equation for $x$:

$3^{x+1} = 5^{2x-1}$

\[
\log_3 (x+1) = \log_5 (2x-1)
\]

\[
(x+1) \log_3 3 = (2x-1) \log_5 5
\]

\[
x \log_3 3 + \log_3 3 = 2x \log_5 5 - \log_5 5
\]

\[
x \log_3 3 - 2x \log_5 5 = -\log_5 5 - \log_3 3
\]

\[
x (\log_3 3 - 2 \log_5 5) = -\log_5 5 - \log_3 3
\]

\[
x = \frac{-\log_5 5 - \log_3 3}{\log_3 3 - 2 \log_5 5}
\]
4. Functions (12 marks)

(a) Determine if \( f(x) = \log(1 - x) \) is a function, state its domain and range sketch its graph.

Each value of \( x \) gives only one value of \( f(x) \), so \( f(x) \) is a function.

**Domain:** We need \( 1 - x > 0 \)

\[ \Rightarrow 1 > x \]

or \( x < 1, \ x \in \mathbb{R} \)

**Range:** \( y \in \mathbb{R} \)

- \( x \)-intercept: \( y = 0 \Rightarrow \log(1 - x) = 0 \Rightarrow 1 - x = 1 \)
  \[ \Rightarrow x = 0 \]

- \( y \)-intercept: \( x = 0 \Rightarrow y = \log(1) = 0 \)
(b) Let \( h(x) = x^2 - 5x + 6 \). Sketch the graph of \( h(x) \), showing all intercepts and the co-ordinates of the vertex. 

\[
\begin{align*}
\text{\( y \)-intercept: } & \quad y = 0 \\
\Rightarrow & \quad x^2 - 5x + 6 = 0 \\
\Rightarrow & \quad (x - 2)(x - 3) = 0 \\
\Rightarrow & \quad x = 2 \text{ or } 3 \\
\text{\( x \)-intercepts: } & \quad x = 0 \Rightarrow y = 6
\end{align*}
\]

\[
\text{\( y \)-intercept: } y = 6
\]

\[
\begin{align*}
\text{Vertex: } & \quad x = \frac{2+3}{2} = \frac{5}{2} \\
& \quad y = \left( \frac{5}{2} \right)^2 - 5 \left( \frac{5}{2} \right) + 6 \\
& \quad = \frac{25}{4} - \frac{25}{2} + 6 \\
& \quad = \frac{25}{4} - \frac{50}{4} + \frac{24}{4} = -\frac{1}{4}
\end{align*}
\]
(c) Let \( y = x^2 - 4x + 6 \).

i. Show that \( y - 2 = (x - 2)^2 \)  

\[
y = x^2 - 4x + 6 \]
\[
= x^2 - 4x + 4 + 2 \quad \text{Take half, square.}
\]
\[
\Rightarrow y - 2 = (x - 2)^2
\]

ii. Hence sketch the graph of \( y \), showing any intercepts and the co-ordinates of the vertex.

\[
y \text{- intercept: } x = 0 \Rightarrow y = 6
\]