1. (a) Show that for LSE estimation:

\[ E = e'e - e'\hat{d} - \hat{d}'c + e'\hat{R}c \]

Hence from the MMSE results of Section 2.2.1 we can show that the normal equations that need to be solved for the \( c = c_{ls} \) that minimises \( E \) are \( \hat{R}c_{ls} = \hat{d} \), that is \( c_{ls} = \hat{R}^{-1}\hat{d} \)

(b) Based on a priori information we may wish to place greater emphasis on different errors, using a weighted LS criterion:

\[ E_w = \sum_{n=0}^{N-1} w(n)\varepsilon(n)^2 = e'W e \quad \text{where} \quad W = \text{diag}\{w(0), w(1), \ldots, w(N-1)\} \]

where we choose small weights where the errors are expected to be large or significant and vice versa. Using the results from (a) show that the weighted LS (WLS) estimator is given by:

\[ c_{wls} = (X'WX)^{-1}X'Wy \]

2. Suppose we have a set of measurements of \( x(n) \) and \( y(n) \) for \( 0 \leq n \leq N - 1 \) with \( N = 5 \) that have been generated by the difference equation:

\[ y(n) = 0.5x(n) + 0.5x(n-1) + v(n), \quad x(-1) = 0 \]

where \( \{v(n)\}_{n=0}^{N-1} = [0, -0.05, 0.05, 0.025, 0] \) is uncorrelated with \( \{x(n)\}_{n=0}^{N-1} = [0, 0.125, 0.25, 0.5, 0.75] \).

(a) Design the second-order optimum MMSE filter to produce estimates of \( y(n) \) given input \( x(n) \). (You can do this by inspection!)

(b) Calculate the second-order LSE FIR filter co-efficients and LSE error variance assuming the full-windowing case. What can you say about the structure of \( \hat{R} \)?

(c) Calculate the second-order LSE FIR filter co-efficients and LSE error variance assuming the no-windowing case. What can you say about the structure of \( \hat{R} \)?

(d) Compare the full-windowing with the no-windowing case.
3. In radar applications the return of a wideband ‘chirp’ signal, \( s(n) \), is observed subject to both narrowband interference (NBI), \( y(n) \), and low-level thermal (white) noise, \( v(n) \), that is:

\[
x(n) = y(n) + s(n) + v(n)
\]

where \( s(n) \), \( y(n) \), and \( v(n) \) are mutually uncorrelated. The ‘chirp’ signal is buried in the NBI and cannot be recovered without removal of the NBI signals. This is carried out by subtracting an estimate of the NBI, \( \hat{y}(n) \), from the observed signal:

\[
x(n) - \hat{y}(n) \equiv s(n) + v(n)
\]

The solution is based on the fact that the ‘chirp’ signal is wideband and has a short correlation length, that is \( r_s(l) = 0 \) for \( |l| \geq D \) for some \( D \): The optimum linear estimate of \( y(n) \) from the data \( x(n) \) is given by:

\[
\hat{y}(n) = c_o^T x(n-D)
\]

Since the required second-order moments are not available an LS estimator is needed

(a) What are the normal equations for LSE estimation that will apply for \( N \) samples of the data? Assume the no-windowing case.

(b) The LSE estimation in (a) requires access to the NBI signal values in the calculation of \( \hat{d} = X^T y \). Restate the LS estimation problem in such a way that the NBI signal values are not needed. \([\text{Hint: Simplify } E\{x(n-k)y(n)\} \text{ so as to remove the dependency on } y(n)\]