Theory of state covariance assignment for linear single-input systems

V.Sreeram
W.-Q.Liu
M.Diab

Indexing terms: Control system design, Covariance assignment, Pole placement

Abstract: An alternative method for state covariance assignment for a general class of linear single-input systems is proposed. New formulas for both discrete and continuous systems are presented based on the inverse solutions of Lyapunov equations. These formulas are similar to the equations for pole placement. The set of all assignable covariance matrices to a single-input system is also characterised. Techniques for constructing these matrices are also described.

1 Introduction

Pole placement and optimal control methods have been popular means of control system design for the past three decades. In both pole placement and optimal control, the main aim is to design linear feedback control laws. In a pole placement (or eigenvalue assignment) problem, the feedback gain matrix is designed so that the closed-loop system has desired poles or eigenvalues. In optimal control, the feedback matrix is chosen to minimise certain performance criteria of the system. As pointed out by Skelton [1], both these techniques can only guarantee that the control system state vector, as a whole, behaves well. Very little can be said about the transient behaviour of the individual state variables. Therefore in practice a system controlled by such methods can have a totally unacceptable transient behaviour.

To overcome these drawbacks the theory of state covariance assignment was first proposed by Skelton and others [2-5]. In the paper by Collins and Skelton [2], a technique for state covariance assignment is proposed for discrete MIMO systems and an algorithm is provided for obtaining state feedback gain matrices. The set of all assignable covariance matrices is also characterised. In a paper by Hotz and Skelton [3] the covariance assignment problem for continuous-time MIMO systems is considered and an algorithm is provided for obtaining state feedback gain matrices. Furthermore, the conditions for covariance controllability are derived. The covariance assignment theory was extended by Skelton and Ikeda [4] to include dynamic controllers.

In a recent paper [6], an alternative technique for state covariance assignment for single-input linear discrete systems of the form

\[ x(k+1) = Ax(k) + b(u(k) + v(k)) \]

where \( v(k) = -gx(k) \) was proposed. The technique was based on the inverse solution of the Lyapunov equation. It was shown that the state covariance assignment problem is related to pole placement problem and the Bass–Gura’s and Ackerman’s formulas for pole placement were extended to state covariance assignment. Furthermore, the set of all assignable state covariance matrices for single-input linear discrete systems was shown to have Toeplitz structure (see also Frazho and Kherat [7]) and thus have some interesting properties. The extensions of these results for covariance assignment of continuous systems was reported by Sreeram and Agathoklis [8] (see also Skelton [9]). However, in these papers, a special class of single-input system wherein the disturbance vector is equal to the input vector was considered. In this paper, a more generalised model for single-input system than the one used in the cited papers is considered. This model has the structure

\[ x(k+1) = Ax(k) + bu(k) + dv(k) \]

where \( u(k) = -gx(k) \) and \( b \neq d \). It is shown that the set of all assignable covariance matrices has a special structure and some interesting properties. Formulas similar to pole placement are derived for covariance assignment. A technique for constructing assignable covariance matrices is also described. Furthermore, all results are extended to continuous-time systems.

2 Preliminaries

In this Section preliminary results on the theory of state covariance assignment are presented. The covariance assignment problem is relatively new and was first proposed in the last decade. In covariance assignment, the objective is to assign a desired state covariance matrix to the closed-loop system.

2.1 State-covariance assignment problem

Consider a single-input discrete system described by

\[ x(k+1) = Ax(k) + bu(k) + dv(k) \]  \hspace{1cm} (1)

where \( u(k) = -gx(k) \), \( x(k) \in \mathbb{R}^n \), \( u(k) \) and \( y(k) \in \mathbb{R} \). The pairs \( \{A, b\} \) and \( \{A, d\} \) are assumed to be controllable. If \( v(k) \) is a zero mean white noise process with variance \( 0 < \Sigma \in \mathbb{R} \) and \( v(k) \) and \( x(0) \) are uncorrelated, the

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IEE Proceedings online no. 19960242

Paper first received 30th May 1995 and in revised form 12th December 1995

The authors are with the Department of Electrical & Electronic Engineering, University of Western Australia, Nedlands, Perth, Australia 6009

steady-state covariance $X$ of the state vector $x$ is given by $X = \mathbb{E}[x(k)x(k)^T]$.  
There are two objectives in the state covariance control problem.  They are

- To find all linear feedback gain vectors that assign a given covariance matrix to the closed-loop system.
- To find the set of all state covariance matrices that can be assigned to a given system by linear state feedback.  For a single input, linear, continuous time system described by
  \[ x(t) = Ax(t) + bu(t) + de(t) \]  
the covariance assignment problem is to find the feedback gain vector $g$ such that the state covariance of the closed-loop system $(A - bg)$,  
\[ X = \lim_{t \to \infty} \mathbb{E} \{x(t)x(t)^T\} \]
achieves a specified value $\tilde{X}$.  The necessary condition for existence of solution to the above problem is that the pairs $(A, b)$ and $(A, d)$ are controllable.  As in the discrete case, this is not sufficient for the existence of a solution.  Furthermore, not all covariance matrices may be assignible to a system by linear state feedback.  Therefore one will be interested in finding set of state covariance matrices that can be assigned to a given system by linear state feedback.

### 2.2 Problem formulation

Consider a single-input system described by eqn. 1.  Since the pairs $(A, b)$ and $(A, d)$ are assumed to be controllable $(A - bg, d)$ is also controllable for any feedback vector $g$.  Note that in general, controllability is neither necessary nor sufficient for covariance assignment.  However, to show that the assignable covariance matrices have a special structure, we assume the controllability of the pairs $(A, b)$ and $(A, d)$.

It is well known that the open-loop system has the state covariance $X$, if it satisfies the following Lyapunov equation:
\[ AXA^T - X = -\mathbb{E}[u(k)u(k)^T] \]  
where $U \in \mathbb{R}^n$ is the variance of the zero mean white noise process, $u(k)$ and $X$ is a positive definite matrix ($X > 0$).  Suppose it is required to assign a state covariance matrix $X > 0$ to the closed-loop system, then it is required to find the stabilising feedback gain $g$ such that it satisfies the following Lyapunov equation:
\[ (A - bg)X(A - bg)^T - X = -\mathbb{E}[u(k)u(k)^T] \]  
As pointed out by Collins and Skelton [2], it is not always possible to find $g$ such that it satisfies the Lyapunov equation.  In this paper, a formula is derived for the set of feedback gains $g$ (if there exists) such that it satisfies the above Lyapunov equation.  Furthermore, a set $(X)$ of state covariance matrices that can be assigned by feedback gain $g$ is also determined.

### 3 State-covariance assignment for discrete systems

In pole placement, the open-loop and closed-loop characteristic polynomials are given and is required to find the feedback vector which yields the desired characteristic polynomial.  In covariance assignment, open-loop and closed-loop covariance matrices are known and is required to find the feedback vector $g$ which yields the desired closed-loop covariance matrix.  Suppose it is possible to find the closed-loop covariance matrix by some means, then the covariance assignment problem becomes a pole placement problem.  The approach followed in this paper is to find the closed-loop polynomial from the closed-loop covariance matrix.  This can be done by solving the Lyapunov eqn. 4 inversely for $A = A - bg$.  The method presented for inverse solution is similar to the method used for the generation of $q$-Markov covers [10].  In the eqn. 4, the matrices $X$ and $\mathbb{E}[u(k)u(k)^T]$ are assumed to be known.  Let $T$ be the transformation [12] which takes $(A, b)$ to controllable canonical form.  Applying this transformation to the Lyapunov eqn. 4 gives $(A, b)$ and $(A - bg, b)$ in controllable canonical form.  The vector $d$ after transformation will take the following form:
\[ d = T^{-1}d = [d_1, d_2, \ldots, d_n]^T \]
The Lyapunov eqn. 4 can now be written as
\[ \bar{A}\bar{X}\bar{A}^T - \bar{X} = -Q = -\mathbb{E}[u(k)u(k)^T] \]
where $\bar{A} = A - bg$.  Expanding this equation gives:
\[
\begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{a}_{n-1} & -\bar{a}_{n-2} & \cdots & -\bar{a}_1 \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn}
\end{bmatrix}
= -\begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
q_{n1} & q_{n2} & \cdots & q_{nn}
\end{bmatrix}
\]
Simplifying as shown in [10], the unknown elements of the closed-loop system matrix can be obtained using the following equations:
\[ \bar{a}_{n-1} = X_{22}^{-1}(X_{1n} - Q_{1n}) - X_{12}\bar{a}_n \]
where
\[ X_{22} = \begin{bmatrix}
x_{22} & x_{23} & \cdots & x_{2n} \\
x_{23} & x_{33} & \cdots & x_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{2n} & x_{3n} & \cdots & x_{nn}
\end{bmatrix} \]
\[ X_{1n} = \begin{bmatrix}
x_{11} & x_{2n} & \cdots & x_{n-1,n}
\end{bmatrix}^T \]
\[ Q_{1n} = \begin{bmatrix}
q_{11} & q_{2n} & \cdots & q_{n-1,n}
\end{bmatrix}^T \]
\[ X_{12} = \begin{bmatrix}
x_{12} & x_{13} & \cdots & x_{1n}
\end{bmatrix}^T \]
\[ \bar{a}_{n-1} = \begin{bmatrix}
-\bar{a}_{n-1} & -\bar{a}_{n-2} & \cdots & -\bar{a}_1
\end{bmatrix}^T \]

Due to the existence of two solutions for this equation, there are in general two solutions to the last row of $A$:
\[ \bar{a}_1 = \begin{bmatrix}
\bar{a}_n \\
\bar{a}_{n-1}
\end{bmatrix}^T X_{22}^{-1}(X_{1n} - Q_{1n}) - X_{12}\bar{a}_n \]
and
\[ \bar{a}_2 = \begin{bmatrix}
-\bar{a}_n \\
\bar{a}_{n-1}
\end{bmatrix}^T X_{22}^{-1}(X_{1n} - Q_{1n}) + X_{12}\bar{a}_n \]
Therefore in general there are two solutions to the characteristic polynomial of the closed-loop system \( (A - bg) \). If the \( a \) is the last row of the system matrix \( A \), the feedback gain vectors are given by
\[
\bar{g}_1 = \bar{a}_1 - a
\]
\[
\bar{g}_2 = \bar{a}_2 - a
\]
If the system is not given in controllable canonical form, the equations for the feedback gain vector can be written
\[
g_1 = (\bar{a}_1 - a)T^{-1}
\]
\[
g_2 = (\bar{a}_2 - a)T^{-1}
\]
Observe that the equations resemble the Bass-Gura formula for pole placement. Hence eqns. 18 and 19 can be called the Bass-Gura formulas for covariance assignment. Now if \( \phi_1(z) \) and \( \phi_2(z) \) are the characteristic polynomials of the closed-loop system for the desired covariance obtained by the foregoing method, the Ackermann’s formula for the covariance assignment can be given as
\[
g_1 = [0 \ 0 \ 0 \ \cdots \ 0 \ 1]M^{-1}\phi_1(A)
\]
\[
g_2 = [0 \ 0 \ 0 \ \cdots \ 0 \ 1]M^{-1}\phi_2(A)
\]
These formulas are not robust numerically. However, like the formulas for pole placement, they help in getting a better understanding of the basic concepts in covariance assignment. Furthermore, these formulas demonstrate the similarities between the pole placement and covariance assignment problems.

Remarks: In eqn. 13, if \( a_0 \neq 0 \), there are two solutions to the closed-loop system matrix
\[
A_1 = A - bg_1
\]
\[
A_2 = A - bg_2
\]
In eqn. 13, if \( a_0 = 0 \) there is a unique solution to the last row of the closed-loop system matrix \( A \) which can be written as
\[
a = \begin{bmatrix}
\bar{a}_n \\
\bar{a}_{n-1} \\
\vdots \\
\bar{a}_1 \\
\bar{a}_0
\end{bmatrix}
\]
Hence the Bass-Gura’s and Ackermann’s formulas for covariance assignment take the following forms:
\[
g = (a - a)W^{-1}M^{-1}
\]
\[
g = [0 \ 0 \ \cdots \ 1]M^{-1}\phi(A)
\]
where \( \phi(A) \) is the characteristic polynomial of the closed-loop system.

3.1 Structure of state covariance matrix
Lemma 1: If the pair \( (A, b) \) is in controllable canonical form, the assignable part of state covariance matrix by linear state feedback is Toeplitz, symmetric, and positive definite. In other words, if \( \bar{X} \) is the closed-loop state covariance matrix assignable by state feedback, \( \bar{X} \) can be decomposed into
\[
\bar{X} = \hat{X} - hh^T
\]
where \( \hat{X} \) is a symmetric positive definite Toeplitz matrix, and
\[
h = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\bar{d}_1 & 0 & 0 & \cdots & 0 & 0 \\
\bar{d}_2 & \bar{d}_1 & 0 & \cdots & 0 & 0 \\
\bar{d}_3 & \bar{d}_2 & \bar{d}_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\bar{d}_{n-1} & \bar{d}_{n-2} & \bar{d}_{n-3} & \cdots & \bar{d}_1 & 0
\end{bmatrix}
\]
Remark: If \( b = d \) in eqn. 1, then \( h = 0 \) and \( \bar{X} = \hat{X} \), as shown by Sreeram and Agathoklis [6].

Lemma 2: For a single-input discrete system (eqn. 1), the entire set of state covariances which may be assigned to the system by linear state feedback gain \( g \) is \( X = \{X : X \) satisfies the equation \( (A - bg)X(A - bg)^T - X = -dUd^T \} \)
\[
X = X^T > 0
\]
\[
X \geq bUb^T
\]
and
\[
\bar{X} = T^{-1}XT^{-T} + hh^T
\]
is symmetric, Toeplitz and positive definite. The proofs for the lemmas are straightforward and are omitted here for brevity. The lemma 2 implies that if the desired covariance \( X \) does not satisfy any of the conditions stated, this covariance is not assignable to the system by linear state feedback.

3.2 Algorithm for covariance assignment
Given a linear stable single-input discrete system (eqn. 1) and a desired closed-loop covariance matrix \( X_d \), the feedback gain vectors \( g_1 \) and \( g_2 \) can be obtained as follows:
(i) Find the transformation matrix \( T \) which takes the system to controllable canonical form.
(ii) Find the new closed-loop covariance matrix using the following transformation:
\[
\bar{X} = T^{-1}X_dT^{-T}
\]
(iii) Check whether \( \bar{X} \) satisfies the conditions stated in lemma 2. If the conditions are not satisfied, no solution exists.
(iv) Compute the last row of new system matrix in controllable canonical form as follows:
\[
a = [0 \ \cdots \ 1]T^{-1}AT
\]
(v) Calculate \( a \) using eqn. 13 and then compute the last rows (\( a_1 \) and \( a_2 \)) of closed-loop system matrices using eqns. 14 and 15, respectively.
(vi) Compute the feedback gain vectors \( g_1 \) and \( g_2 \) using the Bass- Gura’s formulas (eqns. 18 and 19), respectively. The feedback gain vectors can also be obtained via Ackermann’s formulas (eqns. 20 and 21).

3.3 Choosing assignable covariance matrix
We know that in the pole-placement problem one has to choose the locations of the closed-loop poles which can be done easily. In the covariance assignment problem, we need to choose a desirable covariance matrix which can be assignable to the given system. This is not a trivial task. This is because not all positive definite matrices can be assigned to a given system by state feedback. We know from preceding Section that if the system \( (A, b) \) is given in controllable canonical form assignable state covariance is any arbitrarily chosen positive definite Toeplitz matrix. If the system is not given in controllable canonical form the assignable state cov-
arinance is given by
\[ X_i = T (X - hh^T) T^T \]
where \( X \) is a positive definite Toeplitz matrix. So if one can find an arbitrarily positive definite Toeplitz matrix, one can find an assignable state covariance matrix \( X_i \) for the given system. In this Section we describe such a method.

Choosing a positive definite Toeplitz matrix of order 1 is trivial, i.e., one can choose any real number \( x_{1,1} > 0 \). Similarly, choosing a positive definite Toeplitz matrix of order 2 is also trivial: (i) choose any real number \( x_{1,1} > 0 \) and then choose \( x_{1,2} \) s.t.
\[
\begin{bmatrix}
  x_{1,1} & x_{1,2} \\
  x_{1,2} & x_{1,1}
\end{bmatrix} = x_{1,1} - x_{1,2}^2 / x_{1,1} > 0
\]
We now derive below a recursive formula for choosing positive definite Toeplitz matrix of order \( k + 1 \) given a positive definite Toeplitz matrix of order \( k \). This formula can be used recursively for constructing positive definite Toeplitz matrix of order \( n \).

Given a symmetric positive definite Toeplitz matrix \( F \) of order \( k \):
\[
F = \begin{bmatrix}
  x_{1,1} & x_{1,2} & \ldots & x_{1,k-1} \\
  x_{1,2} & x_{1,1} & \ldots & x_{1,k-2} \\
  & & \ldots & \ldots \\
  x_{1,k-1} & x_{1,k-2} & \ldots & x_{1,1}
\end{bmatrix}
\]
we want to find an element \( x \) so that the following matrix of order \( k + 1 \)
\[
\tilde{F} = \begin{bmatrix}
  x_{1,1} & x_{1,2} & \ldots & x_{1,k-1} & x \\
  x_{1,2} & x_{1,1} & \ldots & x_{1,k-2} & x_{1,k-1} \\
  & & \ldots & \ldots & \ldots \\
  x_{1,k-1} & x_{1,k-2} & \ldots & x_{1,1} & x_{1,2} \\
  x & x_{1,1} & \ldots & x_{1,k-2} & x_{1,k-1}
\end{bmatrix}
\]
is also a symmetric positive definite Toeplitz matrix. Since \( F \) is symmetric positive definite, the matrix \( F \) is also symmetric positive definite if and only if
\[
\det \tilde{F} = \det(F) \det(x_{1,1} - w^T F^{-1} w) > 0 \tag{33}
\]
where \( w = \{x, x_{1,k-1}, \ldots, x_{1,1}\} \). Since \( \det(F) > 0 \) in eqn. 33,
\[
x_{1,1} - w^T F^{-1} w > 0 \tag{34}
\]
\[
F^{-1} = \begin{bmatrix}
  P_{1,1} & P_{1,2} \\
  P_{2,1} & P_{2,2}
\end{bmatrix}
\]
and 
\[
z = [x_{1,k-1}, \ldots, x_{1,1}] \]
where \( P_{1,1} \in \mathbb{R}^{k \times k}, P_{1,2} = [p_{1,2}, p_{1,3}, \ldots, p_{1,k-1}] \in \mathbb{R}^{k \times (k-2)}, P_{2,2} \in \mathbb{R}^{k \times (k-2)} \). Then eqn. 34 is equivalent to
\[
x_{1,1} - [x, z] \begin{bmatrix}
  P_{1,1} & P_{1,2} \\
  P_{2,1} & P_{2,2}
\end{bmatrix} \begin{bmatrix}
  x \\
  z
\end{bmatrix} > 0
\]
Simplifying,
\[
P_{1,1} x^2 + 2 (z^T P_{1,2}) x + z^T P_{2,2} z - x_{1,1} < 0 \tag{35}
\]
Since \( P_{1,1} > 0 \) there exists a real \( x \) satisfying eqn. 35 if and only if the discriminant of the quadratic eqn. 35 satisfies the following inequality:
\[
(2 z^T P_{1,2})^2 - 4 P_{1,1} (z^T P_{2,2} z - x_{1,1}) > 0 \tag{36}
\]
Lemma 3 shows that the condition of eqn. 36 holds always. Since eqn. 36 gives two solutions one can construct two positive definite Toeplitz matrices of order \( k + 1 \) from any given positive definite Toeplitz matrix of order \( k \). We can use this result to construct assignable covariance matrix \( F \) of any order.

Lemma 3: If \( F \) is symmetric positive definite,
\[
(z^T P_{1,2})^2 - P_{1,1} (z^T P_{2,2} z - x_{1,1}) = 1 \tag{37}
\]
where all the symbols are defined as before.

Proof: It is easy to see that
\[
F = \begin{bmatrix}
  x_{1,1} & z^T \\
  z & F_1
\end{bmatrix}
\]
Since \( P \) is inverse of \( F \) and the block elements \( P_{i,j}, i, j = 1, 2 \) can be expressed with block elements of \( F \) as follows:

\[
P_{1,1} = (x_{1,1} - z^T F_1^{-1} z)^{-1}
\]
\[
P_{1,2} = (z^T - x_{1,1} F_1)^{-1} z
\]
\[
P_{2,2} = (x_{1,1} (x_{1,1} F_1 - z^T)^{-1})
\]
It is routine to prove the following
\[
(z^T - x_{1,1} F_1)^{-1} = F_1^{-1} - F_1^{-1} z z^T F_1^{-1} \tag{38}
\]
so
\[
z^T P_{1,2} = (Jz)^T (z^T - x_{1,1} F_1)^{-1} z
\]
\[
= z^T J \left[ F_1^{-1} - \frac{F_1^{-1} z z^T F_1^{-1}}{x_{1,1} (x_{1,1} - z^T F_1^{-1} z)} \right] z
\]
\[
= -z^T J F_1^{-1} F_1^{-1} z
\]
\[
= -z^T J F_1^{-1} z
\]
On the other hand,
\[
P_{1,1} = (z^T P_{2,2} z - x_{1,1})
\]
\[
= x_{1,1} \left[ (z^T J F_1^{-1} z x_{1,1} + (z^T J F_1^{-1} z)^2) \right]^{-1}
\]
\[
= (z^T J F_1^{-1} z)^2 \left[ x_{1,1} (x_{1,1} - z^T F_1^{-1} z) \right]
\]
\[
= (x_{1,1} - z^T F_1^{-1} z) = 1
\]
Therefore
\[
(z^T P_{1,2})^2 - P_{1,1} (z^T P_{2,2} z - x_{1,1}) = 1
\]
This completes the proof.

3.4 Example 1
Consider a third-order system described by the state eqn. 1 where
\[
A = \begin{bmatrix}
  -1.8067 & -0.3933 & 4.0400 \\
  2.6133 & 0.7867 & -4.0800 \\
 -0.3867 & -0.2133 & -0.0800
\end{bmatrix}
\]
\[
b = \begin{bmatrix}
  -0.2222 \\
  0.4444
\end{bmatrix}
\]
and \( d = 2.1111 \). Assuming \( U = 1 \), the system has the following state covariance matrix:
\[
\begin{bmatrix}
  I & 0 & 0 \\
  0 & I & 0 \\
  0 & 0 & I
\end{bmatrix}
\]
\[
X = \begin{bmatrix}
237.3401 & -296.7946 & 28.8881 \\
-296.7946 & 384.7260 & -36.1902 \\
28.8881 & -36.1902 & 7.7921
\end{bmatrix}
\]

Let the desired covariance for the closed-loop system be
\[
X_d = \begin{bmatrix}
11.1555 & -2.9474 & 6.4479 \\
-2.9474 & 11.6651 & -0.5695 \\
6.4479 & -0.5695 & 4.3078
\end{bmatrix}
\]

The feedback gain vectors are
\[
g_1 = [0.3780, -0.2070, -2.9790] \\
g_2 = [0.3754, -0.2056, -2.9755]
\]
The corresponding closed-loop system matrices are
\[
A_1 = (A - bg_1) = \begin{bmatrix}
-1.7227 & -0.4393 & 3.3780 \\
2.4453 & 0.8787 & -2.7560 \\
-0.5547 & -0.1213 & 1.2440
\end{bmatrix}
\]
and
\[
A_2 = (A - bg_2) = \begin{bmatrix}
-1.7232 & -0.4390 & 3.3788 \\
2.4465 & 0.8751 & -2.7575 \\
-0.5535 & -0.1219 & 1.2425
\end{bmatrix}
\]
The system matrices can be easily verified to yield the desired closed loop covariance $X_d$.

4 Results for continuous-time systems

In this Section the results of preceding Section are extended to the state covariance assignment of continuous-time systems. The state covariance assignment for continuous-time systems is also based on the inverse solution of Lyapunov equation. There are many techniques available for inverse solution of Lyapunov equation. The method proposed here is similar to the one used in identification and model reduction [11].

Consider the solution of the following continuous-time Lyapunov equation
\[
\dot{X} = AX + XA^T = -Q = -dU^Td^T
\]
where $\dot{X}$ is the desired closed-loop covariance matrix and $A$ is the closed-loop system matrix. Assuming $A$ and $b$ are in controllable canonical form [12], the equation can be expanded as follows:

\[
\begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-\tilde{a}_0 & \tilde{a}_1 & \cdots & -\tilde{a}_{n-1}
\end{bmatrix}
\begin{bmatrix}
x_{1,1} \\
x_{1,2} \\
x_{1,3} \\
\vdots \\
x_{1,n}
\end{bmatrix}
+ \begin{bmatrix}
x_{1,1} \\
x_{1,2} \\
x_{1,3} \\
\vdots \\
x_{1,n}
\end{bmatrix}
= \begin{bmatrix}
x_{1,1} \\
x_{1,2} \\
x_{1,3} \\
\vdots \\
x_{1,n}
\end{bmatrix}
\]

Using the results in [11], it can be shown that
\[
\begin{bmatrix}
x_{1,1} & x_{1,2} & \cdots & x_{1,n}
\end{bmatrix}
= \begin{bmatrix}
x_{1,1} \\
x_{1,2} \\
x_{1,3} \\
\vdots \\
x_{1,n}
\end{bmatrix}
\]

4.1 Structure of state covariance matrix

Lemma 4: If the system $(A, b)$ is in controllable canonical form, the assignable part of state covariance matrix by state feedback is symmetric, positive definite and has signature Hankel [9] or Xiao structure [13]:

\[
\begin{bmatrix}
x_{1,1} & 0 & -x_{2,2} & 0 & x_{3,3} & \cdots \\
0 & x_{2,2} & 0 & -x_{3,3} & 0 & \cdots \\
-x_{2,2} & 0 & x_{3,3} & 0 & -x_{4,4} & \cdots \\
0 & -x_{3,3} & 0 & x_{4,4} & 0 & \cdots \\
x_{3,3} & 0 & -x_{4,4} & 0 & x_{5,5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

(ii) In other words, if $\tilde{X}$ is the desired closed-loop covariance matrix assignable by state feedback, it can be decomposed into $\tilde{X} = X - \tilde{P} - P^T$ where $X$ is a positive definite symmetric Xiao matrix and

\[
\tilde{P} = \sum_{i=0}^{k} (-1)^i P_i P^i
\]

where

\[
\begin{bmatrix}
0 & 0.5q_{1,1} & q_{1,2} & \cdots & q_{1,n-1} \\
0 & 0 & 0.5q_{2,2} & \cdots & q_{2,n-1} \\
0 & 0 & 0 & \cdots & 0.5q_{n-1,n-1}
\end{bmatrix}
\]

and

\[
I = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

Remark: If $b = d$, in eqn. 2, then $\tilde{P} = 0$ and $\tilde{X} = X$ as shown by Sreenam and Gauthoklis [8].

Lemma 5: For a single-input continuous system (eqn. 2), the entire set of state covariances which may be assigned to the system by linear state feedback gain $g$ is $X_n = \{X : X$ satisfies the equation $(A - bg)X + X(A - bg)^T = -dU^Td\}$ where
\[ X = X^T > 0 \] (47)

and
\[ \tilde{X} = T^{-1}XT^{-T} + \tilde{P} + \tilde{P}^T \] (48)
has signature Hankel structure.

The proofs for the lemmas are straightforward and are omitted here for brevity. Lemma 5 implies that if the desired covariance \( X \) does not satisfy any of the conditions stated, this covariance is not assignable to the system by linear state feedback.

4.2 Algorithm for state covariance assignment

Given a linear stable single-input continuous-time system (eqn. 2) and the desired closed-loop covariance matrix \( X_d \), the feedback gain vector \( g \) can be obtained as follows:

(i) Find the transformation matrix \( T \) which takes the system to controllable canonical form.

(ii) Find the new closed-loop covariance matrix using the following transformation:
\[ \tilde{X} = T^{-1}X_dT^{-T} \] (49)

(iii) Check whether \( \tilde{X} \) satisfies the conditions stated in lemma 4. If the conditions are not satisfied, no solution exists.

(iv) Compute the last row of new system matrix in controllable canonical form as follows:
\[ a = [0 \ 0 \ \ldots \ 1]^T T^{-1}AT \] (50)

(v) Compute the last row of the closed-loop system matrix in controllable canonical form using eqn. 42.

(vi) Compute the feedback gain vector \( g \) using the Bass–Gura’s formula (eqn. 45) or Ackermann’s formula (eqn. 46).

4.3 Choosing assignable covariance matrix

As in the discrete case, choosing a desirable covariance matrix assignable to the given system is not a trivial task. We know from preceding Section that if the system \( \{ A, b \} \) is given in controllable canonical form then the assignable state covariance is any positive definite matrix with signature Hankel structure.

\[ \tilde{X} = \begin{bmatrix} x_{1,1} & 0 & -x_{2,2} & 0 & x_{3,3} & \ldots & \ldots \\ 0 & x_{2,2} & 0 & -x_{3,3} & 0 & \ldots & \ldots \\ -x_{2,2} & 0 & x_{3,3} & 0 & x_{4,4} & \ldots & \ldots \\ 0 & -x_{3,3} & 0 & x_{4,4} & 0 & \ldots & \ldots \\ x_{3,3} & 0 & -x_{4,4} & 0 & x_{5,5} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & x_{n,n} \end{bmatrix} \]

If the system realisation is not given in controllable canonical form then the assignable covariance matrix is given by
\[ X_d = T \tilde{X} T^T \] (51)

where \( \tilde{X} \) is positive definite matrix with signature Hankel structure. We now derive a recursive formula for choosing positive definite matrix (with signature Hankel structure) of order \( k \) + 1 given a positive definite matrix (with signature Hankel structure) of order \( k \). This formula can be used for constructing positive definite matrix (with signature Hankel structure) of order \( n \).

To derive the algorithm, we partition a \( k \)th order positive definite matrix (with signature Hankel structure)

\[ \begin{bmatrix} x_{1,1} & 0 & -x_{2,2} & 0 & x_{3,3} & \ldots & \ldots \\ 0 & x_{2,2} & 0 & -x_{3,3} & 0 & \ldots & \ldots \\ -x_{2,2} & 0 & x_{3,3} & 0 & x_{4,4} & \ldots & \ldots \\ 0 & -x_{3,3} & 0 & x_{4,4} & 0 & \ldots & \ldots \\ x_{3,3} & 0 & -x_{4,4} & 0 & x_{5,5} & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & x_{n,n} \end{bmatrix} \]

as follows:
\[ \tilde{X} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \]
where \( A \in \mathbb{R}^{(n-1)\times(n-1)}, \quad B \in \mathbb{R}^{(n-1)\times 1}, \quad C = x_{n,n} \in \mathbb{R} \). We now need the following determinant identity:
\[ \det \tilde{X} = \det(A) \det \left( C - B^T A^{-1} B \right) \]

If \( \tilde{X} \) has to be positive definite, then \( C - B^T A^{-1} B > 0 \) which implies \( x_{n,n} > B^T A^{-1} B \). Since the RHS of the inequality is known, \( x_{n,n} \) can be easily chosen. To choose an assignable covariance matrix, first choose a matrix \( \tilde{X} \) of order 1 and 2, which is trivial. Then use the formula developed recursively to find the elements of a matrix \( X \). The desired assignable covariance matrix \( X_d \) can be then obtained using the transformation of eqn. 51.

4.4 Example 2

Consider a third-order system with system matrices
\[ A = \begin{bmatrix} 0.1818 & 0.5455 & -2.0000 \\ -0.2273 & -0.1818 & 2.0000 \\ -1.7273 & -4.1818 & 4.0000 \end{bmatrix} \]
\[ b = \begin{bmatrix} 0.0779 \\ 0.0260 \\ 0.2597 \end{bmatrix} \]
and \( d = \begin{bmatrix} 1.1948 \\ 2.6494 \end{bmatrix} \)

Assuming \( U = 1 \), the covariance matrix for this system is
\[ X = \begin{bmatrix} 7.5371 & -3.0391 & 0.2132 \\ -3.0391 & 1.5460 & -0.1786 \\ 0.2132 & -0.1786 & 0.9720 \end{bmatrix} \]

Let the desired closed-loop covariance matrix be
\[ X_d = \begin{bmatrix} 0.6110 & -0.4224 & 0.0201 \\ -0.4224 & 0.3772 & -0.0757 \\ 0.0201 & -0.0757 & 0.9671 \end{bmatrix} \]

Using the algorithm proposed the feedback gain vector is
\[ g = [93 \ 118 \ 7] \]

It can be easily verified that the following closed-loop system matrix
\[ \tilde{A} = (A - bg) = \begin{bmatrix} -7.0649 & -8.6494 & -2.5455 \\ -2.1883 & 2.8831 & 2.1818 \\ -25.8831 & -34.8312 & -5.8182 \end{bmatrix} \]

has the desired covariance \( X_d \).

5 Conclusion

In this paper simple formulas have been derived for state covariance assignment. The formulas for feedback gain vectors have been provided for a general class of linear single-input systems. It was shown that there is a similarity between the proposed formulas for state covariance assignment and the Bass–Gura’s and Ackermann’s formulas for pole placement. Furthermore, it has
been shown that the assignable state covariance matrices for both discrete and continuous systems have special structures and can be easily constructed using simple recursive algorithms.

6 Acknowledgment

This work was supported by an individual research grant of the University of Western Australia.

7 References
