CE421-Chapter-2
Computes of System Response

In this discussion we will consider only systems which are linear, continuous and time-invariant.

Solution of homogeneous state equations

We first consider scalar differential equation:

\[ \dot{x}(t) = ax(t) \]  \hspace{1cm} (1)

we assume a solution \( x(t) \) of the form

\[ x(t) = b_0 + b_1 t + b_2 t^2 + \ldots \]  \hspace{1cm} (2)

By substituting this assumed solution into (1) we get

\[ b_1 + 2b_2 t + 3b_2 t^2 + \ldots = a \left( b_0 + b_1 t + b_2 t^2 + \ldots \right) \]  \hspace{1cm} (3)

By equating coefficients of equal powers of \( t \), we get

\[ b_1 = ab_0 \]

\[ b_2 = \frac{1}{2} ab_1 = \frac{1}{2} a^2 b_0 \]

\[ b_3 = \frac{1}{3} ab_2 = \frac{1}{3 \times 2} a^3 b_0 \]
\[ b_k = \frac{1}{k!} a^k b_0 \]

The value of \( b_0 \) is determined by substituting \( t = 0 \) in eqn. (2), or

\[ x(0) = b_0 \]

Hence the solution \( x(t) \) can be written as

\[
x(t) = \left( 1 + at + \frac{1}{2!} a^2 t^2 + \ldots + \frac{1}{k!} a^k t^k + \ldots \right) x(0)
\]

\[ = e^{at} x(0) \]

**Solution of vector-matrix differential equation**

We now consider vector-matrix differential equation:

\[ \dot{x}(t) = Ax(t) \tag{4} \]

By analogy with the scalar case we assume a solution \( x(t) \) is of the form of vector power series in \( t \),

\[ x(t) = B_0 + B_1 t + B_2 t^2 + \ldots \tag{5} \]

By substituting this assumed solution into (4) we get

\[ B_1 + 2B_2 t + 3B_2 t^2 + \ldots = A \left( B_0 + B_1 t + B_2 t^2 + \ldots \right) \]

By equating coefficients of equal powers of \( t \), we get

\[
B_1 = AB_0 \\
B_2 = \frac{1}{2} AB_1 = \frac{1}{2} A^2 B_0 \\
B_3 = \frac{1}{3} AB_2 = \frac{1}{3} \times 2 A^3 B_0 \\
\vdots \quad \vdots \\
B_k = \frac{1}{k!} A^k B_0
\]
The value of $B_0$ is determined by substituting $t = 0$ in eqn. (5), or

$$x(0) = B_0$$

Hence the solution $x(t)$ can be written as

$$x(t) = \left(I + At + \frac{1}{2!} A^2 t^2 + \ldots + \frac{1}{k!} A^k t^k + \ldots\right)x(0)$$

$$= e^{At}x(0)$$

where

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

The infinite series converges for all finite $t$. The matrix

$$\phi(t) = e^{At}$$

is known as the state transition matrix and has the following properties:

1. $\phi(0) = I$.
2. $\phi(t) = [\phi(-t)]^{-1}$ or $\phi^{-1}(t) = \phi(-t)$.
3. $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$.
4. $[\phi(t)]^n = \phi(nt)$.
5. $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0)$.
6. $\frac{d}{dt}(e^{At}) = e^{At}A = Ae^{At}$.

**Exercise:** Prove the above properties of the state-transition matrix.

**Solution of non-homogeneous state equation**

We first consider the scalar case

$$\dot{x}(t) = ax(t) + bu(t)$$

rewriting the above equation

$$\dot{x}(t) - ax(t) = bu(t)$$
multiplying both sides by $e^{-at}$, we obtain

$$e^{-at} [\dot{x}(t) - ax(t)] = \frac{d}{dt} [e^{-at} x(t)] = e^{-at} bu(t)$$

integrating this equation between 0 and $t$, gives

$$e^{-at} x(t) = x(0) + \int_0^t e^{-a\tau} bu(\tau) d\tau$$

or

$$x(t) = e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau$$

we will now consider the vector-matrix non-homogeneous state equation described by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x(t)$ is a $n$-state vector, $A$ is a $n \times n$ matrix and $B$ is a $n \times 1$ vector. By rewriting the above equation we get

$$\dot{x}(t) - Ax(t) = Bu(t)$$

premultiplying both sides by $e^{-At}$, we obtain

$$e^{-At} [\dot{x}(t) - Ax(t)] = \frac{d}{dt} [e^{-At} x(t)] = e^{-At} Bu(t)$$

integrating this equation between 0 and $t$, gives

$$e^{-At} x(t) = x(0) + \int_0^t e^{-A\tau} Bu(\tau) d\tau$$

or

$$x(t) = e^{At} x(0) + e^{At} \int_0^t e^{-A\tau} Bu(\tau) d\tau$$

The output of the system is given by

$$y(t) = Cx(t) = Ce^{At} x(0) + Ce^{At} \int_0^t e^{-A\tau} Bu(\tau) d\tau$$

**Exercise:** Derive the above solutions using Laplace transform approach.
Non-Uniqueness of Set of State Variables

State variables are not unique. For a given system, one can choose an infinite set of state variables to write state equations for the system. Suppose that $x_1, x_2, \ldots, x_n$ are a set of state variables. Thus we may take as another set of state variables, any set of functions

$$
\begin{align*}
\dot{x}_1 &= X_1(x_1, x_2, \ldots, x_n) \\
\dot{x}_2 &= X_2(x_1, x_2, \ldots, x_n) \\
&\vdots &\vdots \\
\dot{x}_n &= X_n(x_1, x_2, \ldots, x_n)
\end{align*}
$$

provided that for every set of values $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$, there corresponds a unique set of values $x_1, x_2, \ldots, x_n$ and vice versa. Thus, if $x$ is a state vector, then $\dot{x}$ given by

$$
\dot{x} = Px
$$

is also a state vector and vice versa.

**Similarity Transformation**

Consider a realization

$$
\begin{align*}
\dot{x} &= Ax + bu \\
y &= cx
\end{align*}
$$

of a system, $G(s)$. Let

$$
\hat{x} = Px \text{ or } x = P^{-1}\hat{x}
$$

where $P$ is nonsingular. Substituting the above transformation in eqn. (6)-(7), we get

$$
\begin{align*}
\dot{x} &= P\dot{x} = PAx + Pbu \\
&= PAP^{-1}\hat{x} + Pbu
\end{align*}
$$

and

$$
y = cx = cP^{-1}\dot{x}
$$
The new realization is given by

\[ \dot{x} = \hat{A} \hat{x} + \hat{bu} \]
\[ y = \hat{c} \hat{x} \]

where

\[ \hat{A} = PAP^{-1} \]
\[ \hat{b} = Pb \]
\[ \hat{c} = cP^{-1} \]

If we choose the transformation as

\[ x = P \hat{x} \]

then the new realization will be

\[ \dot{x} = \hat{A} \hat{x} + \hat{bu} \]
\[ y = \hat{c} \hat{x} \]

where

\[ \hat{A} = P^{-1}AP \]
\[ \hat{b} = P^{-1}b \]
\[ \hat{c} = cP \]

The realizations \( \{A, b, c\} \) and \( \{\hat{A}, \hat{b}, \hat{c}\} \) are similar realizations i.e., they are connected by similarity transformation.

**Properties of similar realizations**

1. **Similar realizations have the same transfer function**

Let

\[ G(s) = c(sI - A)^{-1}b \]

be the transfer function of the realization \( \{A, b, c\} \) and let

\[ \hat{G}(s) = \hat{c}(sI - \hat{A})^{-1}\hat{b} \]
be the transfer function of the realization \( \{ \hat{A}, \hat{b}, \hat{c} \} \). We will now show that

\[
G(s) = \hat{G}(s)
\]

**Proof:**

\[
\hat{G}(s) = \hat{c}(sI - \hat{A})^{-1}\hat{b} \\
= cP(sI - P^{-1}AP)^{-1}P^{-1}b \\
= c\left[P(sI - P^{-1}AP)P^{-1}\right]^{-1}b \\
= c\left[sIPP^{-1} - PP^{-1}APP^{-1}\right]^{-1}b \\
= c(sI - A)^{-1}b
\]

2. **Similar realizations have the same eigenvalues**

Recall that the eigenvalues of a matrix \( A \) are the roots of its characteristic equation:

\[
|\lambda I - A| = 0
\]

To show that similar realizations have the same eigenvalues, we have to show that they have the same characteristic equation, i.e.,

\[
|\lambda I - \hat{A}| = |\lambda I - A|
\]

**Proof:**

\[
|\lambda I - \hat{A}| = |\lambda I - P^{-1}AP| \\
= |P^{-1}(\lambda I - A)P| \\
= |P^{-1}||\lambda I - A||P| \\
= |P^{-1}||P||\lambda I - A| \\
= |\lambda I - A|
\]

**Question:** What is the relation between the poles of the system \( G(s) = c(sI - A)^{-1}b \) and the eigenvalues of the system matrix \( A \)?
Diagonalization

Consider a $n$-th order multi-input, and multi-output system described by

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]  

(8)  

(9)

Assume that $A$ is non-diagonal. Let us define a new state vector such that

\[
x = Mz
\]

(10)

where $M$ is nonsingular. Substituting (3) in (1) and (2) yields

\[
\begin{align*}
\dot{z} &= \hat{A}z + \hat{B}u \\
y &= \hat{C}z + Du
\end{align*}
\]

where

\[
\hat{A} = M^{-1}AM \\
\hat{B} = M^{-1}B \\
\hat{C} = CM
\]

If $M$ can be selected such that $M^{-1}AM$ is diagonal, then $M$ is called the diagonalizing matrix or the modal matrix. If matrix $A$ has $n$ linearly independent eigenvectors then the eigenvector matrix of $A$ can be chosen as the modal matrix.

**Remark:** If matrix $A$ has $n$ distinct eigenvalues then the matrix $A$ has $n$ linearly independent eigenvectors.

If a $n \times n$ matrix $A$ with distinct eigenvalues is given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & -a_1
\end{bmatrix}
\]
then the modal matrix or the eigenvector matrix takes a special form given by

\[
P = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{bmatrix}
\]

The diagonal matrix is given by

\[
\hat{A} = P^{-1}AP = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}
\]

If matrix \( A \) has multiple eigenvalues then diagonalization is impossible because \( P^{-1} \) does not exist. In such cases the transformation matrix of the form:

\[
P = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\lambda_1 & 1 & 0 & \cdots & 0 \\
\lambda_1^2 & 2\lambda_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \frac{d}{d\lambda_1}\lambda_1^{n-1} & \frac{d^2}{d\lambda_1^2}\lambda_1^{n-1} & \cdots & 1
\end{bmatrix}
\]

yields a diagonalized matrix of the following form:

\[
\hat{A} = P^{-1}AP = \begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \lambda_1
\end{bmatrix}
\]

This form is known as the Jordan form.

**Question:** What is the relation between the realization obtained by partial fraction expansion (Chapter 1) and the realization obtained by diagonalization?