LAB II
Control Engineering Laboratory CE 421
Controllability, Observability, Minimality, and Stability

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1 Aim

The aim of this experiment is to study properties of realizations and systems: controllability, observability, minimality and stability.

2 Introduction

Controllability, Observability, and Minimality

Consider an nth order SISO system described by the following realization:

\[ \dot{x} = Ax + bu \quad (1) \]
\[ y = cx + du \quad (2) \]

With respect to the realization, \{A, b, c, d\} (eqns. (1) and (2)) we define controllability, observability, and minimality as follows:

**Controllability** A system realization is said to be completely state controllable if it is possible to transfer the system state from any initial state \( x(t_0) \) to any other desired state \( x(t_f) \) in specified finite time by a control vector \( u(t) \).

**Observability** A system realization is said to be completely observable, if every state \( x(t_0) \) can be completely identified by measurements of the output \( y(t) \) over a finite time.

**Minimality** If the system realization is both controllable and observable then it is said to be minimal.

Note that the state controllability and state observability are properties of the system realizations.

**Conditions for Complete State Controllability**

For system (eqn. (1) and (2)), the controllability matrix is given by

\[ C = \begin{bmatrix} b & Ab & \ldots & A^{n-1}b \end{bmatrix} \]
is a $n \times nr$ matrix. For complete state controllability

$$\text{rank } [C] = n$$

In other words the controllability matrix $C$ should have $n$ linearly independent columns (or rows). Since the controllability matrix is a square matrix, this is equivalent to the condition $\text{det}[C] \neq 0$. For a multi-input system, the matrix $C$ is rectangular of dimension $n \times mn$ where $m$ is the number of inputs. In this case the rank of controllability matrix should be $n$ for complete state controllability.

**Conditions for Complete State Observability**

For system (eqn. (1) and (2)), the observability matrix is given by

$$O = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}$$

For complete state observability

$$\text{rank } [O] = n$$

In other words the observability matrix $O$ should have $n$ linearly independent rows (or columns). Since the observability matrix is a square matrix, this is equivalent to the condition $\text{det}[O] \neq 0$. For a multi-output system, the matrix $O$ is rectangular of dimension $nm \times n$ where $m$ is the number of outputs. In this case the rank of observability matrix should be $n$ for complete state observability.

**Markov Parameters and Hankel Matrices**

Markov parameters are defined as

$$h_i = cA^{i-1}b$$

These parameters are system invariants and can be obtained from the transfer function

$$H(s) = c(sI - A)^{-1}b = \sum_{i=1}^{\infty} h_i s^{-i}$$
An important matrix connected with Markov parameters is

\[
M[i, j] = \begin{bmatrix}
  h_i & h_{i+1} & \cdots & h_{i+j} \\
h_{i+1} & h_{i+2} & \cdots & h_{i+j+1} \\
\vdots & \vdots & & \vdots \\
h_{i+j} & h_{i+j+1} & \cdots & h_{2i} 
\end{bmatrix}
\]

The special cases \( i = 1 \) and \( j = n - 1 \) will encountered often. Matrices such as \( M[i, j] \) that are constant along the anti-diagonals are often called Hankel matrices.

Markov parameters are related to the impulse response of the system. The impulse response of the system is

\[
h(t) = \mathcal{L}^{-1} H(s) = ce^{At}b
\]

The Markov parameters are

\[
cA^tb = h_{i+1} = \frac{d}{dt} h(t)|_{t=0}, \quad i = 0, 1, 2, \ldots
\]

Hankel matrix is defined as follows:

\[
M[1, n - 1] = \begin{bmatrix}
  h_1 & h_2 & \cdots & h_n \\
h_2 & h_2 & \cdots & h_{n+1} \\
\vdots & \vdots & & \vdots \\
h_n & h_{n+1} & \cdots & h_{2n-1}
\end{bmatrix}
\]

\[
= OC
\]

Hankel matrices are used in system identification application. Minimality can be determined by testing for the rank of the Hankel Matrix. If the rank of the Hankel matrix is \( n \) for an \( n \)th order realization, then the realization is said to be minimal.

**Representation of Non-Controllable Realization**

Consider a non-controllable \( n \)-th order SISO realization

\[
\begin{align*}
x &= Ax + bu \\
y &= cx
\end{align*}
\]
Let

$$\text{rank } \mathcal{C}(A, b) = r < n$$

There exists a transformation $T$ such that

$$\tilde{x} = \tilde{A} \tilde{x} + \tilde{b}u$$
$$y = \tilde{c} \tilde{x}$$

where

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_c & \tilde{A}_c \tilde{e} \\ 0 & \tilde{A}_e \end{bmatrix}$$
$$\tilde{b} = T^{-1}b = \begin{bmatrix} \tilde{b}_c \\ 0 \end{bmatrix}$$
$$\tilde{c} = cT = \begin{bmatrix} \tilde{c}_c \\ \tilde{c}_e \end{bmatrix}$$

It can be easily shown that the transformation matrix has the following structure:

$$T = \begin{bmatrix} b & Ab & \ldots & A^{r-1}b & v_1 & v_2 & \ldots & v_{n-r} \end{bmatrix}$$

where the last $n - r$ column vectors $v_1, v_2, \ldots, v_{n-r}$ are chosen such that they are linearly independent of first $r$ columns of the the transformation matrix $T$. Any realization in this form has the following important properties:

1. The $r \times r$ submatrix $\{\tilde{A}_c, \tilde{b}_c, \tilde{c}_e\}$ is controllable

2. The $r \times r$ subsystem has the same transfer function as the original system.

If the state variables $\tilde{x}$ are correspondingly partitioned as

$$\tilde{x} = \begin{bmatrix} \tilde{x}_c \\ \tilde{x}_e \end{bmatrix}$$

then the variables $\tilde{x}_c$ can be said to be controllable and the variables $\tilde{x}_e$ noncontrollable.
Representation of Non-Observable Realization

Consider a non-observable $n$-th order SISO realization

$$\dot{x} = Ax + bu$$
$$y = cx$$

Let

$$\text{rank } \mathcal{O}(A, c) = r < n$$

There exists a transformation $T$ such that

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u$$
$$y = \tilde{c}\tilde{x}$$

where

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} \tilde{A}_o & 0 \\ \tilde{A}_\beta & \tilde{A}_\delta \end{bmatrix}$$

$$\tilde{b} = Tb = \begin{bmatrix} \tilde{b}_o \\ \tilde{b}_\beta \end{bmatrix}$$

$$\tilde{c} = cT^{-1} = \begin{bmatrix} \tilde{c}_o & 0 \end{bmatrix}$$

It can easily shown that the transformation matrix has the following structure:

$$T = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{r-1} \\ v_1 \\ v_2 \\ \vdots \\ v_{n-r} \end{bmatrix}$$

where the last $n - r$ row vectors $v_1, v_2, \ldots, v_{n-r}$ are chosen such that they are linearly independent of first $r$ rows of the the transformation matrix $T$.

Any realization in this form has the following important properties:
1. The $r \times r$ submatrix $\{\bar{A}_o, \bar{b}_o, \bar{c}_o\}$ is observable.

2. The $r \times r$ subsystem has the same transfer function as the original system.

If the state variables $\bar{x}$ are correspondingly partitioned as

$$\bar{x} = \begin{bmatrix} \bar{x}_o \\ \bar{x}_\sigma \end{bmatrix}$$

then the variables $\bar{x}_o$ can be said to be observable and the variables $\bar{x}_\sigma$ nonobservable.

**Lyapunov Stability Analysis**

**Introduction to Stability**

The concept of stability is very important in system theory. In discussing stability, we consider both external and internal stability.

One popular definition of external stability is bounded-input, bounded-output (BIBO) stability which is defined as follows:

**Definition:** A causal system is said to be BIBO stable if the output remains bounded for all bounded inputs.

It is well known that a necessary and sufficient condition for BIBO stability for continuous-time systems is its impulse response is absolutely integrable and for discrete-time system is its impulse response is absolutely summable.

**Definition:** A system realization is internally stable if and only if

- $Re[\lambda_i(A)] < 0$, for continuous-time systems
- $|\lambda_i(A)| < 1$, for discrete-time systems

Internal stability is also known as asymptotic stability. It is well known that external stability does not imply internal stability and internal stability implies external stability. However, for systems described by minimal realizations, both
these stabilities are equivalent. If the characteristic polynomial of the system is known, we can determine stability by using Routh-Hurwitz table for continuous-time systems and Jury’s table for discrete time systems. Another way to determine stability is via Lyapunov stability criterion which is given by the following theorem:

**Theorem:**

The system $\dot{x} = Ax$ is asymptotically stable, i.e., $Re[\lambda_i(A)] < 0$ if and only for any given positive definite symmetric matrix $Q$, there exists a positive definite symmetric matrix $P$ which satisfies:

$$A^T P + PA = -Q$$

The above equation is known as the Lyapunov equation. A number of methods have been proposed for the solution of this equation.

**Remarks**

1. In solving this equation, it is not necessary to choose a positive definite matrix $Q$. Instead you could choose matrix $Q$ to positive semidefinite provided the $\{Q^{1/2}, Q\}$ is observable.

2. If we choose arbitrary positive definite matrix as $Q$ then solve the matrix equation

$$A^T P + PA = -Q$$

to determine $P$, then the positive definiteness of $P$ is necessary and sufficient condition for the asymptotic stability of the system.

3. The final result does not depend on a particular $Q$ matrix chosen so long as it is positive definite (or positive semidefinite as the case may be).

4. To determine the elements of $P$ matrix, we equate matrices, $A^T P + PA$ and $-Q$ element by element. This results in $n(n + 1)/2$ linear equations for the determination of the elements $p_{ij} = p_{ji}$ of $P$.

5. If we denote the eigenvalues of $A$ by $\lambda_1, \lambda_2, \ldots, \lambda_n$ (need not be distinct) then
the Lyapunov equation has a unique solution if and only if

$$\lambda_i + \lambda_j \neq 0$$

This means elements of $P$ are unique. Note that if the matrix $A$ is a stable matrix then $\lambda_i + \lambda_j$ are always non-zero.

6. In determining whether or not there exists a positive definite symmetric matrix $P$, it is convenient to choose $Q = I$, where $I$ is an identity matrix. The elements of $P$ are determined from

$$A^T P + PA = -I$$

and the matrix $P$ is tested for positive definiteness.

3 Procedure

1. Write a function file to compute the controllability matrix of the system.

2. Write a function file to compute the observability matrix of the system.

3. Write a function file to compute the Hankel matrix of the system

4. Write a function file to test the controllability, observability and minimality of the system. Construct some examples and test your programs.

5. Write an m file to decompose a non-controllable realization into controllable and un-controllable subsystems.

6. Write an m file to decompose a non-observable realization into observable and un-observable subsystems.

7. Test the programs created in procedures 5 and 6 by constructing two examples: (i) a fourth order realization with two states controllable and two states uncontrollable, and (ii) a fourth order realization with two states observable and two states unobservable.
8. Investigate the stability of the system described by

\[ \dot{x}(t) = Ax(t) \]

where

\[
(a) \ A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}, \quad (b) \ A = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad (c) \ A = \begin{bmatrix} -1 & -1 \\ 4 & -3 \end{bmatrix}.
\]

using the Lyapunov stability theorem (use matlab function: lyap). The lyap function solves the equation \( A^T P + PA = -Q \), for \( P \) given the matrices \( A \) and \( Q \). For the above examples, check the stability using two different values of \( Q \): (i) a positive definite matrix and (ii) a positive semidefinite matrix. Comment on the results.

9. Verify the results obtained in problems 8 by finding the eigenvalues of the system matrix (use matlab function: eig).

10. Verify the results obtained in problems 8 by first calculating the characteristic polynomial (use matlab function: poly) and then computing its roots (use matlab function: roots).

11. Are the results obtained in problems 9 and 10 the same? Comment.

**Useful functions:** rank - to check the rank of controllability and observability matrices, rand - to generate linearly independent vectors in procedures 5 and 6, and lyap - to solve Lyapunov equation.

### 4 References
