School of Electrical, Electronic and Computer Engineering.

Control Engineering (CE) 447

Tutorial 10

1. Investigate the stability of the system described by

\[ \dot{x}(t) = Ax(t) \]

where

(a) \( A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \),
(b) \( A = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} \), and
(c) \( A = \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix} \)

using the Lyapunov stability theorem (to verify your results in matlab you can use the function lyap).

2. Investigate the stability of the system described by

\[ \dot{x}(k+1) = Ax(k) \]

where

(a) \( A = \begin{bmatrix} -0.2 & 1 \\ 0.5 & 0.7 \end{bmatrix} \),
(b) \( A = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \), and
(c) \( A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} \)

using the Lyapunov stability theorem (to verify your results in matlab you can use the function lyap).
Self Study Questions

1. Explain what is stability?

2. List three ways of determining the stability for linear time invariant continuous and discrete systems.

3. Explain where the poles should lie for stability /instability of (i) continuous time systems and (ii) discrete time systems. Give an example of (i) stable system, and (ii) unstable system for both continuous and discrete systems.

4. Are poles of transfer function $G(s)/G(z)$ same as eigenvalues of system matrix $A$? Why?

5. What is the main advantage of using Routh table for testing the stability of continuous systems.

6. When solving Lyapunov equations (continuous/discrete), how many equations you need to solve simultaneously? In how many unknowns?

7. When is the solution to the Lyapunov equations?

$$A^T P + PA = -Q, \quad A^T PA - P = -Q$$

unique? Since you do not know the eigenvalues of the system matrix $A$ before you test for the stability, how will you know when the conditions for uniqueness is violated?

8. When solving the Lyapunov equations

$$A^T P + PA = -Q, \quad A^T PA - P = -Q$$

for determining the stability, can you choose a positive semi definite $Q$? what conditions does $Q$ has to satisfy? Does the final result (stable/unstable) depend the particular $Q$ matrix chosen?
Tutorial Solutions

Question 1:

a) \( \dot{x}(t) = Ax(t) \)

where

\[
A = \begin{bmatrix}
-0.2 & 1 \\
0.5 & 0.7 \\
\end{bmatrix}
\]

Step 1: choose \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) > 0

Step 2: Solve for \( P \), from the Lyapunov equation

\[
A^TP + PA = -Q \quad \text{or}
\]

\[
\begin{bmatrix}
-2 & -1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22} \\
\end{bmatrix}
+ \begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22} \\
\end{bmatrix}
\begin{bmatrix}
-2 & 1 \\
-1 & 1 \\
\end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
-2p_{11} - p_{12} - 2p_{11} - p_{12} = -1 \Rightarrow 2p_{11} + p_{12} = \frac{1}{2} \quad (1)
\]

\[
-2p_{12} - p_{22} + p_{11} + p_{12} = 0 \Rightarrow p_{11} - p_{12} - p_{22} = 0 \quad (2)
\]

\[
2p_{12} + 2p_{22} = -1 \Rightarrow p_{12} + p_{22} = -\frac{1}{2} \quad (3)
\]

Solving (1), (2) and (3) simultaneously

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & -1 & -1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{12} \\
p_{13} \\
\end{bmatrix}
= \begin{bmatrix}
1/2 \\
0 \\
-1/2 \\
\end{bmatrix}
\]
we have \( p_{11} = -\frac{1}{2}, \quad p_{12} = \frac{3}{2} \) and \( p_{22} = -2 \) \[ \therefore \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -2 \end{bmatrix} \]

Step 3: check for positive definiteness of \( P \)

\[ p_{11} = -\frac{1}{2} < 0 \quad \therefore P \text{ is not positive definite} \]

\[ \therefore \text{ The system } \dot{x}(t) = Ax(t) \text{ is not stable.} \]

Remark: You can verify your result by checking the eigenvalues of \( A \) which are -1.618, and 0.618.

b) \[ \dot{x}(t) = Ax(t) \]

where

\[ A = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} \]

Step 1: choose \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \)

Step 2: Solve for \( P \), from the Lyapunov equation

\[ A^T P + PA = -Q \quad \text{or} \]

\[ \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
\[-2p_{11} + p_{12} = \frac{1}{2}\]  \hspace{1cm} (1)

\[-2p_{12} + p_{22} - 3p_{11} - p_{12} + 2p_{12} = 0\]  \hspace{1cm} (2)

\[-3p_{12} + 2p_{22} = -\frac{1}{2}\]  \hspace{1cm} (3)

Solving (1), (2) and (3) simultaneously

\[
\begin{bmatrix}
2 & 1 & 0 \\
-3 & 0 & 1 \\
0 & -3 & 2
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{12} \\
p_{13}
\end{bmatrix} =
\begin{bmatrix}
-1/2 \\
0 \\
-1/2
\end{bmatrix}
\]

Since coefficient matrix is singular, the solution is nonunique, this implies

\[\lambda_i + \lambda_j = 0\]

Therefore the system is unstable.

c) \[\dot{x}(t) = Ax(t)\]

where

\[A = \begin{bmatrix}
-1 & -1 \\
-1 & -3
\end{bmatrix}\]

Step 1: choose \(Q = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} > 0\)

Step 2: Solve for \(P\), from the Lyapunov equation

\[A^T P + PA = -Q\] for \(P\)
\[
\begin{bmatrix}
-1 & -1 & 0 \\
-1 & -3 & 1 \\
0 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{12} \\
p_{13}
\end{bmatrix}
+ 
\begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix}
\begin{bmatrix}
-1 & -1 \\
-1 & -3
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[-p_{11} - p_{12} = -\frac{1}{2}\]  \(1\)

\[-p_{12} - p_{22} - p_{11} - 3p_{12} = 0 \Rightarrow -p_{11} - 4p_{12} - p_{22} = 0\]  \(2\)

\[-p_{12} - 3p_{22} - p_{12} - 3p_{22} = -1 \Rightarrow p_{12} + 3p_{22} = \frac{1}{2}\]  \(3\)

Solving (1), (2) and (3) simulaltaneously

\[
\begin{bmatrix}
-1 & -1 & 0 \\
-1 & -4 & -1 \\
0 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{12} \\
p_{13}
\end{bmatrix}
= 
\begin{bmatrix}
-1/2 \\
0 \\
1/2
\end{bmatrix}
\]

we have  \(p_{11} = 0.75, \ p_{12} = 0.25, \ \text{and} \ p_{22} = 0.25\)

\[
\begin{bmatrix}
0.75 & -0.25 \\
-0.25 & 0.25
\end{bmatrix}
\]

Step 3: check for positive definiteness of  \(P\)

\[
p_{11} = 0.75 > 0, \quad \begin{vmatrix}
0.75 & -0.25 \\
-0.25 & 0.25
\end{vmatrix} = 0.75 \times 0.25 - (-0.25)^2 > 0
\]

\[
\therefore \ P \text{ is not positive definite}
\]

\[
\therefore \ \text{The system} \ \dot{x}(t) = Ax(t) \ \text{is stable.}
\]
Question 2:

a) Given \( \dot{x}(k+1) = Ax(k) \)

where

\[
A = \begin{bmatrix}
0.2 & 1 \\
0.5 & 0.7
\end{bmatrix}
\]

Step 1: choose \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \)

Step 2: Solve \( A^T PA - P = -Q \) for \( P \)

\[
\begin{bmatrix}
0.2 & 0.5 \\
1.0 & 0.7
\end{bmatrix} \begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix} \begin{bmatrix}
0.2 & 0.5 \\
1.0 & 0.7
\end{bmatrix} - \begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Expanding and simplifying

\[
\begin{bmatrix}
0.96 & 0.2 & 0.25 \\
0.2 & 0.36 & 0.35 \\
1.0 & 1.4 & 0.51
\end{bmatrix} \begin{bmatrix}
p_{11} \\
p_{12} \\
p_{13}
\end{bmatrix} = \begin{bmatrix}
-1 \\
0 \\
-1
\end{bmatrix}
\]

Solving we get \( P = \begin{bmatrix} 0.2686 & -1.5389 \\ -1.5389 & -1.7365 \end{bmatrix} \)

Step 3: check for positive definiteness of \( P \)

\[
p_{11} = 0.2688 > 0, \quad \begin{vmatrix} 0.2688 & -1.5389 \\ -1.5389 & -1.7365 \end{vmatrix} < 0
\]

\( \therefore P \) is not positive definite
The system \( \dot{x}(t) = Ax(t) \) is unstable. (or the system is unstable)

**Remark:** You can verify your results by calculating the eigenvalues

\[ \text{eig}(A) = -0.3 \text{ and } 1.2. \]  
\[ \therefore A \text{ is stable} \]

b) given \( \dot{x}(k+1) = Ax(k) \)

where

\[
A = \begin{bmatrix}
-1 & -2 \\
1 & 1
\end{bmatrix}
\]

Step 1: choose \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \)

Step 2: Solve \( A^T PA - P = -Q \) for \( P \)

\[
\begin{bmatrix}
-1 & -2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix}
\begin{bmatrix}
-1 & -2 \\
1 & 1
\end{bmatrix}
- \begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix}
= -\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Expanding and simplifying

\[
\begin{bmatrix}
0 & -2 & 1 \\
2 & -4 & 1 \\
4 & -4 & 0
\end{bmatrix}
\begin{bmatrix}
p_{11} \\
p_{12} \\
p_{13}
\end{bmatrix}
= \begin{bmatrix}
-1 \\
0 \\
-1
\end{bmatrix}
\]

Since coefficient matrix is singular, the solution is nonunique, this implies

\[ \lambda_i \lambda_j = 1 \]

Therefore the system is unstable.
Remark: You can verify your results by calculating the eigenvalues of $A$ which are -1 and 1.

c) Given \( \dot{x}(k+1) = Ax(k) \)

where

\[
A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}
\]

Step 1: choose \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \)

Step 2: Solve \( A^T PA = -Q \) for \( P \)

\[
\begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Expanding and simplifying

\[
\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1.5 & 1 \\ 0.25 & -1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}
\]

Solving we get \( P = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \)

Step 3: check for positive definiteness of \( P \)

\[
p_{11} = 4 > 0, \quad \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 8 > 0
\]

\( \therefore P \) is positive definite
\[ \dot{x}(t) = Ax(t) \]
is stable. (or the system is stable)

1) System is stable when bounded input produces bounded output.

2) Stability tests.

<table>
<thead>
<tr>
<th>Continuous system</th>
<th>Discrete system</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Routh test</td>
<td>Jury’s test</td>
</tr>
<tr>
<td>b) Lyapunov stability theorem</td>
<td>c) Checking the eigen values of system matrix.</td>
</tr>
</tbody>
</table>

3) Example

\[
\begin{align*}
\dot{x} &= Ax \\
x(k+1) &= Ax(k)
\end{align*}
\]

where

\[
\begin{align*}
A &= \begin{bmatrix}
-1 & 2 \\
0 & -2
\end{bmatrix} \text{ stable system} \\
A &= \begin{bmatrix}
0.8 & 0 \\
0 & 0.6
\end{bmatrix} \text{ stable system} \\
A &= \begin{bmatrix}
-1 & 2 \\
0 & -2
\end{bmatrix} \text{ unstable system} \\
A &= \begin{bmatrix}
1.2 & 0.2 \\
0 & 0.7
\end{bmatrix} \text{ unstable system}
\end{align*}
\]
4) Poles of \( G(s) = \) eigenvalues of \((A)\)

Poles of \( G(z) = \) eigenvalues of \((A)\)

\[
G(s) = c(sI - A)^{-1}b \\
G(z) = c(zI - A)^{-1}b
\]

\[
\frac{c \text{ Adj}(sI - A)b}{sI - A} = \frac{c \text{ Adj}(zI - A)b}{zI - A}
\]

Roots of \( |sI - A| = 0 \) are eigenvalues of \( A \) and are also poles of \( G(s) \)

Roots of \( |zI - A| = 0 \) are eigenvalues of \( A \) and are also poles of \( G(z) \)

5) **Advantage**

You can determine the stability without actually computing the roots of a polynomial which is numerically ill-conditioned problem.

6) \( n(n+1)/z \) equations in \( n(n+1)/z \) unknowns. This is because the matrices \( P \) and \( Q \) are symmetric.

7) Solution of Lyapunov equation

\[
A^TP + PA = -Q
\]

is unique iff \( \lambda_i + \lambda_j \neq 0, \ i = 1,2,\ldots,n \) and \( j = 1,2,\ldots,n \)

where \( \lambda_i \) are eigenvalues of \( A \). Solution of Lyapunov equation

\[
A^TPA - P = -Q \quad \text{is unique iff} \quad \lambda_i \lambda_j \neq 1, \ \text{where} \ \lambda_i \ \text{are eigenvalues of} \ A.
\]

8) Yes, you can choose a positive semi definite matrix for \( Q \) provided the following rank condition is satisfied.

\[
\text{rank} \begin{bmatrix} Q^{1/2} & 0 & \cdots & 0 \\ Q^{1/2}A & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ Q^{1/2}A^{n-1} & \cdots & \cdots & Q^{1/2}A^{n-1} \end{bmatrix} = n
\]

The final result (stability/instability) does not depend upon the particular \( Q \) matrix chosen.