1. The linearized equations of a free pendulum are $\dot{\theta} + \omega_c^2 \theta = u$. Show that output feedback $-k\theta$ will not stabilize the system but that this can be don’t using a system $(s + \alpha)/(s + \beta)$ in the feedback path, provided $\alpha < \beta$.

2. Consider a second order system described the following state space equations:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
$$

$$
y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix}
$$

a) Is the system controllable?

b) Is the system observable?

c) Find a feedback gain vector, $k$ to place the system poles at $\{-1, -2\}$, if possible.

d) Repeat c) to place poles at $\{-2, -2\}$.

e) Explain any discrepancy in answers to (c) and (d).

f) Which state variables should be measured to stabilize the system using feedback.

g) Find the transfer function when $c_1 = 1$ and $c_2 = 0$.

h) Repeat g) for $c_1 = 0$ and $c_2 = 1$.

3. A helicopter near hover can be described by the equations

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
-0.02 & -1.4 & 9.8 \\
-0.01 & -0.4 & 0.0 \\
0.0 & 1.0 & 0.0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ 
\begin{bmatrix}
9.8 \\
6.3 \\
0.0
\end{bmatrix} u
$$
where $x_1 =$ horizontal velocity, $x_2 =$ pitch rate, $x_3 =$ pitch angle,

$u =$ rotor tilt angle. (a) Find the open loop poles (b) Show that a state feedback law to move the poles to $s = -2, s = -1 \pm j$ is

$$k = \begin{bmatrix} 0.0628 & 0.4706 & 0.9949 \end{bmatrix}$$

4. Consider the system defined by

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By using the state feedback control $u = -Kx,$ is desired to have the closed-loop poles at $s = -2 \pm j4, s = -10.$ Determine the state feedback gain matrix $K.$
Question 1:

The transfer function

\[ \frac{1}{s^2 + \omega_0^2} \]

with output feedback

\[ -k \]

The transfer function is

\[ \frac{1}{s^2 + \omega_0^2 + k} \]

The poles are at

\[ \begin{cases} 
\pm j \sqrt{\omega_0^2 + k} & k \geq \omega_0^2 \\
\pm j \sqrt{k - \omega_0^2} & k < \omega_0^2 
\end{cases} \]

Clearly the system is unstable. Therefore the output feedback will not stabilize the system.
The closed loop transfer function is given by

\[
\frac{1}{s^2 + \omega_0^2} = \frac{1}{1 + \left(\frac{s + \alpha}{s + \beta}\right) \left(\frac{1}{s^2 + \omega_0^2}\right)} = \frac{(s + \beta)}{(s + \beta) \left(s^2 + \omega_0^2\right) + (s + \alpha)}
\]

\[
= \frac{s + \beta}{s^2 + \beta s^2 + \omega_0^2 s + \omega_0^2 \beta + s + \alpha} = \frac{s + \beta}{s^3 + s^2 \beta + (\omega_0^2 + 1)s + \omega_0^2 \beta + \alpha}
\]

Routh Table

\[
\begin{array}{ccc}
s^3 & 1 & \omega_0^2 + 1 \\
s^2 & \beta & \omega_0^2 \beta + \alpha \\
s & \beta \omega_0^2 + \beta - \left(\omega_0^2 \beta + \alpha\right) & \beta \\
s & \beta - \alpha & \beta
\end{array}
\]

For stability of the closed loop system \( \beta - \alpha > 0 \) or \( \beta > \alpha \)

**Question 2:**

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & 2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
c_1 & c_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]
a) \( C = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \) \( \text{rank } [C] = 1 \), not controllable.

By inspection of state space equation mode \( \lambda = -1 \) is not controllable.

b) \( \Phi = \begin{bmatrix} c_1 & c_2 \\ -c_1 & 2c_2 \end{bmatrix} \)

\[ |\Phi| = 2c_1c_2 + c_1c_2 = 3c_1c_2 \]

\[ [\Phi] \neq 0 \Rightarrow c_1 \text{ and } c_2 \neq 0 \text{ for observability.} \]

c) \( |sI - A + bk| = \begin{vmatrix} s+1 & 0 \\ 0 & s-2 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} [k_1 \ k_2] \)

\[ = \begin{vmatrix} s+1 & 0 \\ k_1 & s-2+k_2 \end{vmatrix} = (s+1)(s-2+k_2) \]

desired polynomial = \( (s+1)(s+2) \)

Equating we get \( k_2 = 4 \) or \( K = \begin{bmatrix} 0 \\ k_2 \end{bmatrix} \)

d)

\[ |sI - A + bk| = \begin{vmatrix} s+1 & 0 \\ 0 & s-2 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} [k_1 \ k_2] \)

\[ = \begin{vmatrix} s+1 & 0 \\ k_1 & s-2+k_2 \end{vmatrix} = (s+1)(s-2+k_2) \]

\[ = (s+2)(s+2) \]

It is not possible to find \( k \) to place poles at \( s = -2, -2 \)
e) This is because the mode corresponding to eigenvalue $s = -1$ is not controllable.

f) The state variables $x_2$ should be measured to stabilize the system using feedback.

g) \[ G(s) = c(sI - A)^{-1}b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} s+1 \\ 0 \\ s-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} (s+1)^{-1} \\ 0 \\ (s-2)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ (s-2)^{-1} \end{bmatrix} = 0 \]

h) \[ G(s) = c(sI - A)^{-1}b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} s+1 \\ 0 \\ s-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} (s+1)^{-1} \\ 0 \\ (s-2)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ (s-2)^{-1} \end{bmatrix} = \frac{1}{s-2} \]

**Question 3:**

The desired closed-loop characteristic polynomial

\[ \alpha(s) = (s+2)(s+1-j)(s+1-j) \]
\[ \alpha(s) = (s + 2)(s^2 + 2s + 2) = (s^3 + 4s^2 + 6s + 4) \]

\[ |sI - A + bk| = \begin{bmatrix}
  s & 0 & 0 \\
  0 & s & 0 \\
  0 & 0 & s
\end{bmatrix}
\begin{bmatrix}
  -0.02 & -1.4 & 9.3 \\
  -0.01 & -0.4 & 0 \\
  0 & 1.0 & 0
\end{bmatrix}
+ \begin{bmatrix}
  9.8 \\
  6.3 \\
  0
\end{bmatrix}
[k_1 \ k_2 \ k_3]
\]

\[ = \begin{bmatrix}
  s + 0.02 & 1.4 & -9.8 \\
  0.01 & s + 0.4 & 0 \\
  0 & -1.0 & s
\end{bmatrix}
+ \begin{bmatrix}
  9.8k_1 \\
  6.3k_1 \\
  0
\end{bmatrix}
\]

\[ = \begin{bmatrix}
  9.8k_2 \\
  6.3k_2 \\
  6.3k_3
\end{bmatrix}
\]

where \( k = [0.0628 \ 0.4706 \ 0.9949] \)

\[ |sI - A + bk| = s^3 + 4s^2 + 6s + 4 \] (verify this using matlab).

**Question 4:**

The desired characteristic equation for the closed loop is

\[ \alpha(s) = (s + 2 - j4)(s + 2 + j4)(s + 10) \]

\[ = (s^2 + 4s + 20)(s + 10) = s^3 + 14s^2 + 60s + 200 \]

Let the desired feedback gain matrix

\[ k = [k_1 \ k_2 \ k_3] \]

The closed-loop characteristic polynomial is given by
\[ |sI - A + bk| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 0 \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \]

\[ = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 + k_1 & s + k_2 & s + 6 + k_3 \end{bmatrix} \]

\[ |sI - A + bk| = s^3 + (6 + k_3)s^2 + (5 + k_2)s + (1 + k_1) \]

Equating \[ |sI - A + bk| = \alpha(s) \]

we get

\[ 6 + k_3 = 14, \quad 5 + k_2 = 60 \quad \text{and} \quad 1 + k_1 = 200 \]

Solving we get

\[ k_1 = 199, \quad k_2 = 55, \quad k_3 = 8 \]

\[ k = \begin{bmatrix} 199 \\ 55 \\ 8 \end{bmatrix} \]