LAB II
Control Engineering Laboratory CE 447
Discretisation of Continuous-Time State Equations

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1 Aim

The aim of this experiment is to study the discretization of continuous-time state equations

2 Introduction

If we wish to compute the state $x(t)$ of the following linear system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= cx(t)
\end{align*}
\](1)

by use of a digital computer, we must convert a continuous-time state equation to a discrete-time state equation. In the following, we shall present such a procedure. We assume that the input is piecewise constant; that is the input $u(t)$ changes values only at discrete instants of time. Inputs of this type occur in sampled-data systems or in systems in which digital computers are used to generate the input signal, $u(t)$. A piece-wise constant function is often generated by a sampler and a filter, called zero-order hold, as shown in the figure:

Let

\[
u(t) = u(k) \text{ for } kT \leq t < (k + 1)T; \ k = 0, 1, 2, \ldots
\](3)

where $T$ is a positive constant, called the sampling period. The discrete times $0, T, 2T, \ldots$ are called sampling instants. The behaviour of the system (1)-(2)
with piecewise constant inputs given in eqn. (3) can be computed using the following:

\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau
\]

\[
y(t) = Cx(t)
\]

However, if only behaviour at sampling instants \(0, T, 2T, \ldots\), is of interest, a discrete-time dynamic equation can be written to give the response of \(x(k) = x(kT)\) at \(k = 0, 1, 2, \ldots\). From (4) we have

\[
x(k + 1) = e^{A(k+1)T}x_0 + \int_0^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau
\]

\[
= e^{AT} \left[ e^{AkT}x_0 + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau \right] + \int_{kT}^{(k+1)T} e^{A(kT+T-\tau)}Bu(\tau)d\tau
\]

The term in brackets in the above equation is equal to \(x(k)\); the input \(u(\tau)\) is constant in the interval \((kT, kT + T)\) and is equal to \(u(k)\); hence the above equation becomes, after the change of variable \(\alpha = kT + T - \tau\).

\[
x(k + 1) = e^{AT}x(k) + \left( \int_0^T e^{A\alpha}d\alpha \right) Bu(k)
\]

which is a discrete-time state equation. Therefore, if the input is piecewise constant over the same interval \(T\), and if only the response at the sampling instants is of interest, the dynamical equation (1)-(2) can be replaced by the following discrete-time linear time-invariant dynamical equation:

\[
x(k + 1) = \hat{A}x(k) + \hat{B}u(k)
\]

\[
y(k) = \hat{C}x(k)
\]

where

\[
\hat{A} = e^{AT}
\]

\[
\hat{B} = \left( \int_0^T e^{A\tau}d\tau \right) B
\]

\[
\hat{C} = C
\]
3 Procedure

1. Write function file to discretise a linear time-invariant, continuous-time system. The first line of your function file should be the following form:

   function [ad,bd,cd,dd] = discretise(ac,bc,cc,dc,ts)

   where \( \{ac, bc, cc, dc\} \): continuous-time system (input to the function)
   \( \{ad, bd, cd, dd\} \): discrete-time system (output of the function)
   ts: sampling period

2. Discretise the following system with \( ts = 0.1, 0.5, 1.0sec \).

   \[
   A = \begin{bmatrix}
   0 & 0 & -2 \\
   1 & 0 & -5 \\
   0 & 1 & -4 \\
   \end{bmatrix}
   \]

   \[
   b = \begin{bmatrix}
   1 \\
   0 \\
   0 \\
   \end{bmatrix}, \quad c = \begin{bmatrix}
   8 & -26 & 66 \\
   \end{bmatrix}
   \]

3. Derive the actual step response of the system using the solution of state-space equations, we derived in Notes-1. Plot the actual step response of the system for 0 to 10 seconds using small increments of 0.1 second.

4. Compute the step response of the system using the discretised state equations. Plot the step response using the command "stairs".

5. Compare the step response of the system obtained using the discretised model with the actual step response of the system and comment.

6. In your report, show also that the discretized system matrices

   \[
   \hat{A} = e^{AT} \quad \text{and} \quad \hat{B} = \left( \int_0^T e^{Ax} dx \right) B
   \]

   can be computed via the computation of the following matrix:

   \[
   e^{A B} + \begin{bmatrix}
   A & B \\
   0 & 0 \\
   \end{bmatrix}^T
   \]