Controllability A system realization is said to be completely state controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any other desired state $x(t_f)$ in specified finite time by a control vector $u(t)$.

Observability A system realization is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurements of the output $y(t)$ over a finite time.

Remark 1 A system which is not completely observable implies that some of its state variables are shielded from observation.

Remark 2 Note that the state controllability and state observability are properties of the system realizations.

Minimality If the system realization is both controllable and observable then it is said to be minimal.

Example 1

Consider the following diagonal realization:

$$\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\vdots \\
\dot{z}_n
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
& & \ddots \\
0 & 0 & \lambda_n
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{bmatrix} + \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix} u$$

$$y = \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{bmatrix}$$

The above realization is controllable iff $b_i \neq 0$ for $i = 1, 2, \ldots, n$.

The above realization is observable iff $c_i \neq 0$ for $i = 1, 2, \ldots, n$. 
Example 2

Consider the following Jordan form realization

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\vdots \\
\dot{z}_{n-1} \\
\dot{z}_n
\end{bmatrix}
= \begin{bmatrix}
\lambda_1 & 1 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 \\
& & \ddots & \ddots \\
0 & 0 & \ddots & 1 \\
0 & 0 & \cdots & \lambda_1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{n-1} \\
z_n
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_{n-1} \\
b_n
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
c_1 & c_2 & \cdots & c_{n-1} & c_n
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{n-1} \\
z_n
\end{bmatrix}
\]

The above realization is controllable iff \( b_n \neq 0 \) and is observable iff \( c_1 \neq 0 \).

Linear independence of vectors

The vectors \( x_1, x_2, \ldots, x_n \) are said to be linearly independent if

\[
c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0
\]

where \( c_1, c_2, \ldots, c_n \) are constants, implies

\[
c_1 = c_2 = c_3 = \ldots = c_n = 0
\]

Conversely, the vectors \( x_1, x_2, \ldots, x_n \) are said to be linearly dependent if and only if \( x_i \) can be expressed as a linear combination of \( x_j \) (\( j = 1, 2, \ldots, n; j \neq i \)) or

\[
x_i = \sum_{j=1}^{n} c_j x_j
\]

for some set of constants \( c_j \).

Rank of a matrix

A matrix \( A \) is called of rank \( m \) if the maximum number of linearly independent rows (or columns) is \( m \). Hence, if there exists a \( m \times m \) submatrix \( M \) of \( A \) such that \( |M| \neq 0 \), and the determinant of every \( r \times r \) submatrix (where \( r \geq m + 1 \)) of \( A \) is zero, then the rank of \( A \) is \( m \).
Complete State Controllability of Continuous Systems

In this section we derive the conditions for complete state controllability. Here we do not assume any special structure for system matrices as we did when we derived conditions for controllability and observability for diagonal and Jordan realizations.

Consider the continuous time system given by

\[
\dot{x} = Ax + bu
\]

where \( x \) is a \( n \)-state vector, \( u \) is a control signal, \( A \) is a \( n \times n \) matrix and \( b \) is a \( n \times 1 \) matrix.

We assume that the initial time is \( t_0 = 0^- \) and the final time \( t_f = 0^+ \). We also assume that the input to be a linear combination of impulse function and its derivatives:

\[
u(t) = b_1 \delta(t) + b_2 \delta^{(1)}(t) + \ldots b_n \delta^{(n-1)}(t)
\]

Here we will show that the system (1) can be taken from any initial state \( x(0^-) \) to any final state \( x(0^+) \) in finite time by suitable input. Since the input used here is of impulse nature, the final state can be reached in ”zero” time. Note that inputs of this nature are impractical. However, I am using this type of input to simplify the derivation of controllability. Controllability condition can also be derived using non-impulsive inputs and the derivation is a bit more complicated.

We know that the solution of eqn. (1) is

\[
x(t) = e^{At}x(0) + \int_{0^-}^{t} e^{A(t-\tau)}bu(\tau)d\tau
\]

Applying the definition of complete state controllability, we have

\[
x(0^+) = x(0^-) + \int_{0^-}^{0^+} e^{-A\tau}bu(\tau)d\tau
\]
Simplifying we get

\[ x(0+) = x(0-) + \begin{bmatrix} b & Ab & \ldots & A^{n-1}b \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \]

(2)

It is clear from the above equation that if the matrix

\[ C = \begin{bmatrix} b & Ab & \ldots & A^{n-1}b \end{bmatrix} \]

is nonsingular, the final state \( x(0+) \) can be reached instantaneously from any initial state \( x(0-) \). On the other hand, if the matrix \( C \) is singular, no matter what the constants, \( b_i \)'s are certain vectors \( x(0+) - x(0-) \) can not be obtained using the impulsive input defined earlier. Therefore, for complete state controllability, the the matrix \( C \) has to be nonsingular. This matrix is known as the controllability matrix.

**Remark** The above result can be easily extended to multi-input continuous as well as discrete systems. Consider the multi-input systems given by

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ x(k+1) = Ax(k) + Bu(k) \]

where \( u \) is a \( r \)-vector and the controllability matrix

\[ C = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \]

is a \( n \times nr \) matrix. For complete state controllability

\[ \text{rank } [C] = n \]

In other words the controllability matrix \( C \) should have \( n \) linearly independent columns (or rows).
Conditions for Complete State Observability

Consider the system described by

\[ \dot{x}(t) = Ax(t) + bu(t) \]

we can write

\[ y(t) = cx(t) \]
\[ \dot{y}(t) = c\dot{x}(t) = cAx(t) + cbu(t) \]
\[ \ddot{y}(t) = cA\dot{x}(t) + cb\dot{u}(t) \]
\[ = cA^2x(t) + cAbu(t) + cb\dot{u}(t) \]

and so on which can be conveniently arranged in matrix form as say

\[ \mathcal{Y}(t) = \mathcal{O}x(t) + \mathcal{T}\mathcal{U}(t) \]

where

\[ \mathcal{Y}(t) = \begin{bmatrix} y(t) & \dot{y}(t) & \ldots & y^{n-1}(t) \end{bmatrix}^T \]
\[ \mathcal{U}(t) = \begin{bmatrix} u(t) & \dot{u}(t) & \ldots & u^{n-1}(t) \end{bmatrix}^T \]

\[ \mathcal{O} = \mathcal{O}(c, A) = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} \]

and \( \mathcal{T} \) is a lower triangular Toeplitz matrix with a first column

\[ \begin{bmatrix} 0 \\ cb \\ cAb \\ \vdots \\ cA^{n-2}b \end{bmatrix} \]

Assuming that \( \mathcal{U}(0) = 0 \) then we have

\[ \mathcal{Y}(0) = \mathcal{O}x(0) \]
and the question is whether we can find an initial state vector \( x(0) \) for a given vector \( \mathcal{Y}(0) \). If \( \mathcal{O} \) has all \( n \) columns linearly independent, i.e., if the \( n \times n \) matrix \( \mathcal{O} \) is nonsingular, then we can always find \( x(0) \) for any \( n \)-vector \( \mathcal{Y}(0) \) as \( x(0) = \mathcal{O}^{-1}\mathcal{Y}(0) \).

On the other hand if \( \mathcal{O} \) is singular (not of full rank) we can find a solution \( x(0) \) for a special choices of \( \mathcal{Y}(0) \) namely those that lie in the column range space of \( \mathcal{O} \).

In order to have full freedom in the choice of \( x(0) \), the observability matrix \( \mathcal{O}(c, A) \) should be nonsingular. Such realizations are said to be observable.

Suppose we have a physical system described by the state-space equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t) \\
y(t) &= cx(t) \\
x(0) &= x_0
\end{align*}
\]

We assume that we know the matrices \( \{A, b, c\} \) and also the input and output functions \( \{u(t), t \geq 0\}, \{y(t), t \geq 0\} \). The problem is to determine the states \( \{x(t), t \geq 0\} \). Now this problem is very close to initial condition problem because really the only unknown in our problem is the initial state \( x_0 \) knowing \( x_0, \{A, b, c\} \) and \( \{u(t), t \geq 0\} \) we could set up the equation

\[
\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0
\]

and thus obtain \( x(t) \) as a function of \( t \).

**Extension to multi-output continuous and discrete cases**

If the system has multiple outputs (\( r \) outputs) then the observability matrix

\[
\mathcal{O} = \begin{bmatrix}
C \\
CA \\
:\ \\
CA^{n-1}
\end{bmatrix}
\]

should be of rank \( n \).
Theorem 1 If a transfer function
\[ H(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} \]
has one controllable and observable \(n\)-th order realization then all \(n\)-th order realizations must also be controllable and observable.

Theorem 2 A realization \(\{A, b, c\}\) is minimal if and only if \(a(s) = \text{det}(sI - A)\) and \(b(s) = c\text{Adj}(sI - A)b\) are relatively prime.

Theorem 3 Any two minimal realizations can be connected by a unique similarity transformation.

Markov Parameters The Markov parameters are defined as
\[ h_i = cA^{i-1}b \]
These parameters are system invariants and can be obtained from the transfer function
\[ H(s) = c(sI - A)^{-1}b = \sum_{i=1}^{\infty} h_is^{-i} \]
An important matrix connected with Markov parameters is
\[ M[i,j] = \begin{bmatrix} h_i & h_{i+1} & \ldots & h_{i+j} \\ h_{i+1} & h_{i+2} & \ldots & h_{i+j+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i+j} & h_{i+j+1} & \ldots & h_{i+2j} \end{bmatrix} \]
The special cases \(i = 1\) and \(j = n - 1\) will encountered often. Matrices such as \(M[i,j]\) that are constant along the anti-diagonals are often called Hankel matrices.

Markov parameters are related to the impulse response of the system. The impulse response of the system is
\[ h(t) = \mathcal{L}^{-1}H(s) = ce^{At}b \]
The Markov parameters are
\[ cA^ib = h_{i+1} = \frac{d^i}{dt^i} h(t)|_{t=0}, \quad i = 0, 1, 2, \ldots \]
Hankel matrix is defined as follows:

\[
M_{[1,n-1]} = \begin{bmatrix}
  h_1 & h_2 & \ldots & h_n \\
  h_2 & h_2 & \ldots & h_{n+1} \\
  \vdots & \vdots & & \vdots \\
  h_n & h_{n+1} & \ldots & h_{2n-1}
\end{bmatrix}
= OC
\]

Hankel matrices are used in system identification application.

**Representation of Non-Controllable Realization**

Consider a non-controllable \(n\)-th order realization

\[
\dot{x} = Ax + bu \\
y = cx
\]

Let

\[
\text{rank } C(A,b) = r < n
\]

There exists a transformation \(T\) such that

\[
\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \\
y = \bar{c}\bar{x}
\]

where

\[
\bar{A} = T^{-1}AT = \begin{bmatrix}
  \bar{A}_c & \bar{A}_{ce} \\
  0 & \bar{A}_e
\end{bmatrix}
\]

\[
\bar{b} = T^{-1}b = \begin{bmatrix}
  \bar{b}_c \\
  0
\end{bmatrix}
\]

\[
\bar{c} = \begin{bmatrix}
  \bar{c}_c & \bar{c}_e
\end{bmatrix}
\]

Any realization in this form has the following important properties:

1. The \(r \times r\) submatrix \(\{\bar{A}_c, \bar{b}_c, \bar{c}_e\}\) is controllable
2. The \(r \times r\) subsystem has the same transfer function as the original system.
If the state variables $\bar{x}$ are correspondingly partitioned as

$$\bar{x} = \begin{bmatrix} \bar{x}_c \\ \bar{x}_e \end{bmatrix}$$

then the variables $\bar{x}_c$ can be said to be controllable and the variables $\bar{x}_e$ noncontrollable.

The realization $\{\bar{A}, \bar{b}, \bar{c}\}$ can be depicted as shown in the figure:

**Representation of Non-Observable Realization**

Consider a non-observable $n$-th order realization

$$\dot{x} = Ax + bu \quad y = cx$$

Let

$$\text{rank } O(A, b) = r < n$$

There exists a transformation $T$ such that

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \quad \bar{y} = \bar{c}\bar{x}$$

where

$$\bar{A} = T^{-1}AT = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{\bar{o}} & \bar{A}_\bar{o} \end{bmatrix}$$

$$\bar{b} = T^{-1}b = \begin{bmatrix} \bar{b}_o \\ \bar{b}_\bar{o} \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} \bar{c}_o & 0 \end{bmatrix}$$
It is easy to show that the above realization has the following important properties:

1. The $r \times r$ submatrix $\{\bar{A}_o, \bar{b}_o, \bar{c}_o\}$ is observable

2. The $r \times r$ subsystem has the same transfer function as the original system.

If the state variables $\bar{x}$ are correspondingly partitioned as

$$\bar{x} = \begin{bmatrix} \bar{x}_o \\ \bar{x}_o \end{bmatrix}$$

then the variables $\bar{x}_o$ can be said to be observable and the variables $\bar{x}_o$ nonobservable.

The realization $\{\bar{A}, \bar{b}, \bar{c}\}$ can be depicted as shown in the figure:

**Obtaining a minimal realization**

Given a non-minimal realization described by the state equation

$$\begin{align*}
\dot{x} &= Ax + bu \\
y &= cx
\end{align*}$$

One can find an invertible transformation that will transform the given realization to the following form

$$\begin{align*}
\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{b}u \\
y &= \bar{c}\bar{x}
\end{align*}$$
where

\[ \bar{x} = \begin{bmatrix} \bar{x}_{c,o} \\ \bar{x}_{c,\bar{o}} \\ \bar{x}_{\bar{c},o} \\ \bar{x}_{\bar{c},\bar{o}} \end{bmatrix} \]

\[ \bar{A} = \begin{bmatrix} \bar{A}_{c,o} & 0 & \bar{A}_{1,3} & 0 \\ \bar{A}_{2,1} & \bar{A}_{c,\bar{o}} & \bar{A}_{2,3} & \bar{A}_{2,4} \\ 0 & 0 & \bar{A}_{\bar{c},o} & 0 \\ 0 & 0 & \bar{A}_{4,3} & \bar{A}_{\bar{c},\bar{o}} \end{bmatrix} \]

\[ \bar{b} = \begin{bmatrix} \bar{b}_{c,o} \\ \bar{b}_{c,\bar{o}} \\ 0 \\ 0 \end{bmatrix} \]

\[ \bar{c} = \begin{bmatrix} \bar{c}_{c,o} & 0 & \bar{c}_{\bar{c},o} & 0 \end{bmatrix} \]

Remarks

1. The subsystem \( \{ \bar{A}_{c,o}, \bar{b}_{c,o}, \bar{c}_{c,o} \} \) is controllable and observable.

2. The subsystem

\[ \left\{ \begin{bmatrix} \bar{A}_{c,o} \\ \bar{A}_{2,1} \end{bmatrix}, \begin{bmatrix} \bar{b}_{c,o} \\ \bar{b}_{c,\bar{o}} \end{bmatrix}, \begin{bmatrix} \bar{c}_{c,o} & 0 \end{bmatrix} \right\} \]

is controllable.

The subsystem

\[ \left\{ \begin{bmatrix} \bar{A}_{c,o} & \bar{A}_{1,3} \\ 0 & \bar{A}_{\bar{c},o} \end{bmatrix}, \begin{bmatrix} \bar{b}_{c,o} \\ 0 \end{bmatrix}, \begin{bmatrix} \bar{c}_{c,o} & \bar{c}_{\bar{c},o} \end{bmatrix} \right\} \]

is observable.

3. The subsystem \( \{ \bar{A}_{c,\bar{o}}, 0, 0 \} \) is uncontrollable and unobservable.

4. The transfer function of the subsystem \( \{ \bar{A}_{c,o}, \bar{b}_{c,o}, \bar{c}_{c,o} \} \) is same as the transfer function of the original system \( \{ \bar{A}, \bar{b}, \bar{c} \} \).
The realization \( \{\bar{A}, \bar{b}, \bar{c}\} \) can be be depicted as shown in the following figure:

**Some Practical Examples**

**Example 1: Satellite trajectory control** [1]

Consider a satellite in a circular orbit at an altitude of 250 nautical miles above the earth, as illustrated in the figure below:

The satellite motion (in the orbit plane) is described by the normalized state variable
model:

\[ \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_r + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_t \]

where the state vector \( x \) represents normalized perturbations from the circular, equatorial orbit, \( u_r \) is the input from a radial thruster, \( u_t \) is the input from a tangential thruster, and \( \omega = 0.0011 \text{rad/s} \) (approximately one orbit of 90 minutes) is the orbital rate for the satellite at the specific altitude. In the absence of perturbations, the satellite will remain in the nominal circular, equatorial orbit. However, disturbances such as aerodynamic drag can cause the satellite to deviate from its nominal path. The problem is to design a controller that commands the satellite thrusters in such a manner that the actual orbit remains near the desired circular orbit. Before commencing with the design, we always check controllability. In this case we investigate controllability using the radial and tangential thrusters independently.

Suppose the tangential thruster fails (i.e., \( u_t = 0 \)), and only the radial thruster is operational. Is the satellite controllable from \( u_r \) only? Suppose if radial thruster fails (\( u_r = 0 \)) and only the tangential thruster is operational, is the satellite controllable from \( u_t \) only?

Matlab Programs

radial.m

% This script computes the satellite controllability with
% a radial thruster only (i.e., failed tangential thruster)
%
w = 0.0011;
A = [0 1 0 0; 3*w^2 0 0 2*w; 0 0 0 1; 0 -2*w 0 0];
b1 = [0;1;0;0];
Pc = ctrb(A,b1);
n = det(Pc);
if abs(n) < eps
    disp('Satellite is uncontrollable with radial thruster only!')
else
    disp('Satellite is controllable with radial thruster only!')
end

tangent.m
%
% This script computes the satellite controllability with
% a tangential thruster only (i.e., failed radial thruster)
%
w = 0.0011;
A = [0 1 0 0; 3*w^2 0 0 2*w; 0 0 0 1; 0 -2*w 0 0];
b2 = [0;1;0;0];
Pc = ctrb(A,b2);
n = det(Pc);
if abs(n) < eps
    disp('Satellite is uncontrollable with tangential thruster only!')
else
    disp('Satellite is controllable with tangential thruster only!')
end
Example 2: A vertical takeoff and landing (VTOL) aircraft [1]

A linearized model of a VTOL aircraft is

\[ \dot{x} = Ax + B_1u_1 + B_2u_2 \]

where

\[
A = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 0.3681 & -0.7070 & 1.4200 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

and

\[
B_1 = \begin{bmatrix}
0.4422 \\
3.5446 \\
-5.5200 \\
0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.1761 \\
-7.5922 \\
4.4900 \\
0
\end{bmatrix}
\]

The state vector components are (i) \( x_1 \) is the horizontal velocity (knots) (ii) \( x_2 \) is the vertical velocity (knots) (iii) \( x_3 \) is the pitch rate (degrees/second), and (iv) \( x_4 \) is the pitch angle (degrees). The input \( u_1 \) is used mainly to control the vertical motion, and \( u_2 \) is for the horizontal motion.

(a) Compute the eigenvalues of the system matrix \( A \). Is the system stable?

(b) Determine the characteristic polynomial associated with \( A \) using the \textit{poly} function.

(c) Compute the roots of the characteristic equation and compare with the eigenvalues in part (a).

(d) Is the system controllable from \( u_1 \) alone? What about from \( u_2 \) alone?
Example 3: Cart with Inverted Pendulums [2]

A cart of mass $M$ has two inverted pendulums on it of lengths $l_1$ and $l_2$, both with bobs of mass $m$. For small $|\theta_1|$ and $|\theta_2|$, the equations of motion are:

\[
M \ddot{v} = -mg\theta_1 - mg\theta_2 + u \\
\quad m \left( \dot{v} + l_i \ddot{\theta}_i \right) = mg\theta_i, \quad i = 1, 2
\]

where $v$ is the velocity of the cart and $u$ is an external force applied to the cart (see the figure).

(a) Is it always possible to “control” both pendulums, i.e., keep them both vertical by using the input $u$?

(b) Is the system observable with output $y = \theta_1$?

Example 4: Dynamics of a Hot Air Balloon [2]

Approximate equations of motion for a hot air balloon are

\[
\dot{\theta} = -\frac{1}{\tau_1} \theta + u \\
\dot{\theta} = -\frac{1}{\tau_2} v + \sigma \theta + \frac{1}{\tau_2} w \\
\dot{h} = v
\]

where $\theta$ is the temperature change of air balloon away from the equilibrium temperature, $u$ is proportional to change in heat added to air in balloon (control), $v$ is the
vertical velocity, $h$ is the change in altitude from equilibrium altitude, and $w$ is the vertical wind velocity (disturbance).

(a) Can the temperature change $\theta$ and a constant wind velocity $w$ be observed by a continuous measurement of altitude change $h$? (Assume that $u$ is known)

(b) Determine the transfer function from $u$ to $h$ and from $w$ to $h$. Is the system completely controllable by $u$? Is it completely controllable by $w$?

**Example 5:** Consider the network [3] shown in the figure:

The input is a current source and the output is the voltage across the $2 - \Omega$ resistor shown. The voltage $x$ across the capacitor is the only state variable of the network. If $x(0) = 0$, no matter what input is applied, because of the symmetry of the four resistors, voltage across the capacitor is always zero. Therefore, it is not possible to transfer the capacitor voltage, $x(0) = 0$ to any non-zero $x$. Thus, the state variable equation describing the network is not controllable. If $x(0) = 0$ is different from zero, then it will generate a response across the output $y$. Thus, it is possible that the equation is observable. Write the state-space equations for the system and then investigate controllability and observability.

**Example 6:** Consider the network [3] shown in the figure:

The input is a current source and the output $y$ is the voltage across the resistor. The network has two state variables: the voltage $x_1$ across the capacitor and the current $x_2$ through the inductor. Non-zero $x_1(0)$ and/or $x_2(0)$ will excite a response inside the LC loop. However, the current $i(t)$ always equals $u(t)$ and the output always equals $2u(t)$ no matter what $x_1(0)$ and $x_2(0)$ are. Therefore, there is no way to determine the
initial state from $u(t)$ and $y(t)$. Thus the state equation describing the network will not be observable. Because LC loop is connected directly to the input, it is possible that the realization is controllable. Write the state-space equations for the system and then investigate controllability and observability.

**Example 7:** In the previous example, if $y$ is the voltage across the inductor instead of resistor, is the new system realization controllable and observable?

---

**Causes of Uncontrollability and Unobservability [4]**

**Pole-Zero Cancellation**

One of the causes of uncontrollability or unobservability is pole zero cancellation. Consider the second-order system given by

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The transfer function is given by

$$G(s) = \frac{c_1(s + 2)b_1 + c_2(s + 1)b_2}{(s + 1)(s + 2)}$$

Now, if $b_1 = 0$ (or $c_1 = 0$), the mode at -1 becomes uncontrollable (or unobservable) and pole term $(s + 1)$ gets cancelled in the transfer function. Similarly, when $b_2 = 0$ (or $c_2 = 0$), the mode at -2 becomes uncontrollable (or unobservable) and the pole term
(s + 2) will get cancelled out. This example demonstrates that either uncontrollability or unobservability (or both) implies pole-zero cancellation in the transfer function. The pole-zero cancellation in a transfer function implies either uncontrollability or unobservability (or both).

**Redundancy in State Variables**

Another cause of uncontrollability or unobservability is the introduction of redundant or unnecessary state variables. Consider the simple RL circuit shown in the figure.

![RL Circuit Diagram](image)

The circuit input is a voltage source and the output is the current flowing through the inductor. From circuit theory, we know that

\[ u(t) = i(t) + \frac{di(t)}{dt} \]

If we let \( x = i(t) \), and \( y(t) = i(t) \), then we get

\[
\begin{align*}
\dot{x} &= -x + u \\
y &= x
\end{align*}
\]

The transfer function of this system is given by

\[ G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s + 1} \]
Suppose electrical charge is selected also as a state variable, then
\[
\begin{align*}
x_1 &= q \\
x_2 &= \dot{q} = i \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 + u \\
y &= x_2
\end{align*}
\]

The state-space equations are:
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

where
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

The transfer function is now given by
\[
G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \frac{s}{s(s + 1)} = \frac{1}{s + 1}
\]

The observability matrix of this second order realization is
\[
O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}
\]

The rank of this matrix is 1 indicating that one of the state variables is not observable. This does not mean that the circuit is unobservable. It simply means that the model is not good and in fact, the extra state variable, charge is redundant for modelling purposes.

**Inappropriate or Insufficient Control Actuators or Sensors**

An appropriate model and sufficient number of actuators (for control action) and sensors (for observing state variables) are needed to achieve control objectives in the design of systems. To demonstrate this idea, consider the problem of stabilizing an inverted pendulum on a moving cart shown in figure:
Suppose the objective is to balance the pendulum and stop the cart. A linearized model for the above is given by

\[
\dot{x} = \begin{bmatrix}
0 & -a & 0 \\
0 & 0 & 1 \\
0 & a & 0 \\
\end{bmatrix} x + \begin{bmatrix}
b_1 \\
0 \\
-b_2 \\
\end{bmatrix} u
\]

The state variables are cart velocity, pendulum angle, and pendulum angular velocity. To meet the objective, one of the state variables is measured and fed back. This feedback signal is used to drive a motor which will move the cart back and forth to stabilize the system. Note that asymptotic stability implies that all states will approach zero, which means that the pendulum will be balanced and the cart will stop moving. We can show easily that this system is controllable by checking the rank of the controllability matrix:

\[
C = \begin{bmatrix}
b_1 & ab_1 & -ab_2 \\
0 & -b_2 & 0 \\
-b_2 & 0 & -ab_2 \\
\end{bmatrix}
\]

Now the question is where to place the sensor, or which state should be measured so that the overall system is observable. If we measure the pendulum angle and use that as the feedback signal, we get

\[
y = \begin{bmatrix}
0 & 1 & 0 \\
\end{bmatrix} x
\]

This system is not observable in this case because

\[
|O| = \begin{vmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & a & 0 \\
\end{vmatrix} = 0
\]
Intuitively, we can know that it is possible to balance the pendulum while the cart is still moving. Therefore, using pendulum angle as the feedback signal would not be a good choice. Similar arguments can be made against measuring the pendulum angular velocity. Finally, using the cart velocity as the measured signal,

\[ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \]

we can verify that the system is observable because

\[ |O| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{vmatrix} = a^2 \neq 0 \]

Therefore, it is theoretically possible to meet the design objectives by measuring the cart velocity.

Note that when the system complexity increases, we may not have the benefit of our intuition anymore, and we have to resort to system concepts.
Lyapunov Stability Analysis

Introduction to Stability

The concept of stability is very important in system theory. In discussing stability, we consider both external and internal stability.

One popular definition of external stability is bounded-input, bounded-output (BIBO) stability which is defined as follows:

**Definition:**

A causal system is said to be BIBO stable if the output remains bounded for all bounded inputs.

It is well known that a necessary and sufficient condition for BIBO stability for continuous-time systems is its impulse response is absolutely integrable and for discrete-time system is its impulse response is absolutely summable.

**Definition:** A system realization is internally stable if and only if

- $\text{Re}[\lambda_i(A)] < 0$, for continuous-time systems
- $|\lambda_i(A)| < 1$, for discrete-time systems

Internal stability is also known as asymptotic stability. It is well known that external stability does not imply internal stability and internal stability implies external stability. However, for systems described by minimal realizations, both these stabilities are equivalent. If the characteristic polynomial of the system is known, we can determine stability by using Routh-Hurwitz table for continuous-time systems and Jury’s table for discrete time systems.
Routh Stability Criterion: A Review

This criterion tells us whether or not there are positive roots in a polynomial without actually solving for the roots. Therefore this criterion is used in control system to determine the stability of the system.

The procedure for Routh’s Stability Criterion is as follows:

1. Write the polynomial in $s$ in the following form:

$$a_0s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n = 0$$

where the coefficients are real numbers.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots on the $j\omega$ axis or there are positive real roots. Therefore in such a case the system is not stable. If we are only interested in the absolute stability, there is no need to proceed further. If we are interested in finding the number of roots on the right half of $s$-plane (or roots with positive real parts) then

3. Construct the following table or array:

<table>
<thead>
<tr>
<th>$s^n$</th>
<th>$s^{n-1}$</th>
<th>$s^{n-2}$</th>
<th>$s^{n-3}$</th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>$a_1$</td>
<td>$b_1$</td>
<td>$c_1$</td>
<td>$d_1$</td>
<td>$e_1$</td>
<td>$f_1$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$b_2$</td>
<td>$c_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_4$</td>
<td>$a_5$</td>
<td>$b_3$</td>
<td>$c_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_6$</td>
<td>$a_7$</td>
<td></td>
<td>$c_4$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

where the coefficients $b_1$, $b_2$, and so on are computed as follows:

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$$

$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$$

$$\vdots$$
The coefficients \( c_1, c_2, \) and so on are computed as follows:

\[
\begin{align*}
  c_1 &= 
  \frac{b_1a_3 - a_1b_2}{b_1} \\
  c_2 &= 
  \frac{b_1a_5 - a_1b_3}{b_1} \\
  &\vdots 
\end{align*}
\]

This row is continued till the \( n \)th row has been completed. The complete array of coefficients is triangular. Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculations; without altering the stability criterion.

Routh’s stability criterion states that the number of roots of the characteristic polynomial with positive real roots is equal to the number of changes in sign of the coefficients of the first column of the array.

The necessary and sufficient condition that all roots lie in the left half of \( s \)-plane is that all the coefficients of the polynomial are positive and all the elements of the first column of the array are also positive.

**Remark** The stability of discrete-time systems can be determined using Jury’s stability criterion.

**Scalar Functions**

Before we study Lyapunov stability analysis, we review the material on the scalar functions, which is useful in understanding the stability concepts.

Scalar functions can be classified as follows:

- positive definiteness
- negative definiteness
- positive semidefiniteness
- negative semidefiniteness
- indefiniteness
Scalar functions that play an important role in the Lyapunov stability analysis is the quadratic form:

\[ V(x) = x^T P x = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{12} & p_{22} & \ldots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \ldots & p_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \]

\( x \) is a real vector and \( P \) is a symmetric matrix.

**Sylvester’s Criterion**

A quadratic form \( v(x) = x^T P x \) is positive definite if all the successive principal minors of \( P \) are positive; that is

\[
p_{11} > 0, \quad \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0, \ldots, \quad \begin{vmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{12} & p_{22} & \ldots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \ldots & p_{nn} \end{vmatrix} > 0
\]

\( v(x) = x^T P x \) is positive semidefinite if all the principal minors are non-negative. \( v(x) = x^T P x \) is negative definite if \(-v(x)\) is positive definite. Similarly, \( v(x) \) is negative semidefinite if \(-v(x)\) is positive semidefinite. If \( v(x) \) is none of the above, then it is indefinite.

**Remark:** You can also determine whether \( P \) is positive definite, negative definite, . . . . . from the eigenvalues of the matrix.

**Example 2** Show that the following quadratic form

\[ v(x) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - x_2x_3 - 4x_1x_3 \]

is positive definite
Lyapunov Stability

Another way to determine stability is via Lyapunov stability criterion which is given by the following theorem:

**Theorem:**

The system $\dot{x} = Ax$ is asymptotically stable, i.e., $\text{Re}[\lambda_i(A)] < 0$ if and only for any given positive definite symmetric matrix $Q$, there exists a positive definite symmetric matrix $P$ which satisfies:

$$A^T P + P A = -Q$$

The above equation is known as the Lyapunov equation. A number of methods have been proposed for the solution of this equation.

**Remarks**

1. In solving this equation, it is not necessary to choose a positive definite matrix $Q$. Instead you could choose matrix $Q$ to positive semidefinite provided the $\{Q^{1/2}, A\}$ is observable.

2. If we choose arbitrary positive definite matrix as $Q$ then solve the matrix equation

$$A^T P + P A = -Q$$

to determine $P$, then the positive definiteness of $P$ is necessary and sufficient condition for the asymptotic stability of the system.

3. The final result does not depend on a particular $Q$ matrix chosen so long as it is positive definite (or positive semidefinite as the case may be).

4. To determine the elements of $P$ matrix, we equate matrices, $A^T P + P A$ and $-Q$ element by element. This results in $n(n + 1)/2$ linear equations for the determination of the elements $p_{ij} = p_{ji}$ of $P$.

5. If we denote the eigenvalues of $A$ by $\lambda_1, \lambda_2, \ldots, \lambda_n$ (need not be distinct) then the Lyapunov equation has a unique solution if and only if

$$\lambda_i + \lambda_j \neq 0$$
This means elements of $P$ are unique. Note that if the matrix $A$ is a stable matrix then $\lambda_i + \lambda_j$ are always non-zero.

6. In determining whether or not there exists a positive definite symmetric matrix $P$, it is convenient to choose $Q = I$, where $I$ is an identity matrix. The elements of $P$ are determined from

$$A^T P + PA = -I$$

and the matrix $P$ is tested for positive definiteness.

**Example 3**

Consider the second order system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the stability of the system.

**Lyapunov Stability Analysis of Discrete-Time Systems**

**Theorem** Consider the discrete-time system

$$x(k+1) = Gx(k)$$

where $x$ is a state vector ($n$-vector) and $G$ is a $n \times n$ constant non-singular matrix. A necessary and sufficient condition for asymptotic stability of the system is that given any positive definite real symmetric matrix $Q$, there exists a positive definite and real symmetric matrix $P$ such that

$$G^T PG - P = -Q$$

**Remark:** The solution to discrete Lyapunov equation is unique if and only if

$$\lambda_i [G] \cdot \lambda_j [G^T] \neq 1 \ \forall i, j$$

where $\lambda_i [G]$ is $i$-th eigenvalue of the system matrix $G$. 
Effect of Discretization on Stability

We now consider the stability of a discrete-time system obtained by discretizing a continuous-time system. Consider a continuous-time system given by

\[ \dot{x} = Ax \]

and the corresponding discrete-time system

\[ x((k+1)T) = Gx(kT) \]

where

\[ G = e^{AT} \]

If the continuous-time system is asymptotically stable, that is if the eigenvalues of a matrix \( A \) have negative real parts then

\[ \|G\|^n \to 0 \text{ as } n \to \infty \]

and the discretized system is also asymptotically stable. That is because if \( \lambda_i \)'s are the eigenvalues of \( A \) then \( e^{\lambda_i T} \)'s are the eigenvalues of \( G \) (Note that \( |e^{\lambda_i T}| < 1 \) if \( \lambda_i T \) is negative).

**Example** Consider the following system

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+2)
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    -0.5 & -1
\end{bmatrix} \begin{bmatrix}
    x_1(k) \\
    x_2(k)
\end{bmatrix}
\]

Determine the stability of the system.
Effect of Discretization on Controllability and Observability

When a continuous-time system (having complex conjugate poles), is discretized, the controllability and observability of the resulting discretized system may be impaired by certain choice of sampling periods due to pole-zero cancellation. If the original system realization \( \{A, B, C\} \) is completely state controllable and observable, the resulting discretized system realization is also completely state controllable and observable if and only if for every eigenvalue of \( A \) the relation

\[
\text{Re}\{\lambda_i\} = \text{Re}\{\lambda_j\}
\]

implies

\[
\text{Im} (\lambda_i - \lambda_j) \neq \frac{2n\pi}{T}
\]

where \( T \) is the sampling period and \( n = \pm 1, \pm 2, \ldots \). Note that when the original system, \( A \) has only real poles, the pole-zero cancellation will not occur during discretization.

References:


