1. Let $x(n)$ be a stationary process with zero mean and autocorrelation $r_x(k)$. We form the process, $y(n)$, as follows:

$$y(n) = x(n) + f(n)$$

where $f(n)$ is a known deterministic sequence. Find the mean $m_y(n)$ and the autocorrelation $r_y(k,l)$ of the process $y(n)$.

2. A discrete-time random process $x(n)$ is generated as follows:

$$x(n) = \sum_{k=1}^{p} a(k)x(n-k) + w(n)$$

where $w(n)$ is a white noise process with variance $\sigma_w^2$. Another process, $z(n)$ is formed by adding noise to $x(n)$,

$$z(n) = x(n) + v(n)$$

where $v(n)$ is white noise with a variance of $\sigma_v^2$ that is uncorrelated with $w(n)$. (a) Find the power spectrum of $x(n)$. (b) Find the power spectrum of $z(n)$.

3. Suppose we are given a linear shift-invariant system having a system function $H(z)$ that is excited by zero mean exponentialy correlated noise $x(n)$ with an autocorrelation sequence:

$$r_x(k) = \left(\frac{1}{2}\right)^{|k|}$$

Let $y(n)$ be the output process, $y(n) = x(n) * h(n)$ (a) Find the power spectrum, $P_y(z)$, of $y(n)$. (b) Find the autocorrelation, $r_y(k)$, of $y(n)$. (c) Find the cross-correlation, $r_{xy}(k)$, between $x(n)$ and $y(n)$. (d) Find the cross-power spectral density, $P_{xy}(z)$, which is the z transform of the cross-correlation $r_{xy}(k)$. 

4. Find the power spectrum for each of the following wide-sense stationary random processes that have the given autocorrelation sequences.

(a) \( r_x(k) = 2\delta(k) + j\delta(k - 1) - j\delta(k + 1) \)

(b) \( r_x(k) = \delta(k) + 2(0.5)^{|k|} \)

(c) \( r_x(k) = 2\delta(k) + \cos(\pi k/4) \)

5. Find the autocorrelation sequence corresponding to each of the following power spectral densities.

(a) \( P_x(e^{j\omega}) = 3 + 2\cos\omega \)

(b) \( P_x(e^{j\omega}) = \frac{4}{5 + 3\cos\omega} \)

(c) \( P_x(z) = \frac{-2z^2 + 5z - 2}{3z^2 + 10z + 3} \)

6. Show the following properties of power density spectrum:

(a) \( S_{XX}(-\Omega) = S_{XX}(\Omega) \) when \( X(t) \) is real

(b) \( S_{XX}(\Omega) = \Omega^2 S_{XX}(\Omega) \)

that is the power density spectrum of the derivative, \( dX(t)/dt \) is \( \Omega^2 \) times the power density spectrum of \( X(t) \).

7. Show the following properties of cross power density spectrum:

(a) \( S_{XY}(\Omega) = S_{YX}(-\Omega) = S_{YX}^*(\Omega) \)

(b) \( \text{Re}[S_{XY}(\Omega)] \) is an even function of \( \Omega \)

(c) \( \text{Im}[S_{XY}(\Omega)] \) is an odd function of \( \Omega \)

8. A wide-sense stationary random process \( X(t) \) with autocorrelation

\[ R_{XX}(\tau) = Ae^{-|\tau|} \]

where \( A \) and \( a \) are real positive constants is applied to the input of a LTI system with impulse response

\[ h(t) = e^{-bt}u(t) \]

where \( b \) is a real positive constant. Find the autocorrelation of the output \( Y(t) \) of the system.
Solutions

1. The mean of the process is given by

\[ m_y(n) = E\{y(n)\} = E\{x(n)\} + f(n) = f(n) \]

The autocorrelation of the process is given by

\[ R_{XX}(k, l) = E\{y(k)y(l)\} = E\{[x(k) + f(k)][x(l) + f(l)]\} = E\{x(k)x(l)\} + f(k)f(l) = R_{XX}(k, l) + f(k)f(l) \]

2. In this problem a white noise process is input to a AR model having the transfer function:

\[ H(z) = \frac{1}{A(z)} \]

where

\[ A(z) = 1 - \sum_{k=1}^{p} a(k)z^{-k} \]

The power spectrum of the output process is given by

\[ S_{XX}(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega})W(e^{j\omega}) = \frac{\sigma_w^2}{|A(e^{j\omega})|^2} \]

The process \( x(n) \) is given by

\[ z(n) = x(n) + v(n) = \sum_{k=-\infty}^{\infty} h(k)w(n-k) + v(n) \]

Since \( v(n) \) is uncorrelated with \( w(n) \), it is uncorrelated with \( x(n) \). Therefore, we have

\[ R_{ZZ}(k) = R_{XX}(k) + R_{VV}(k) \]

\[ S_{ZZ}(e^{j\omega}) = S_{XX}(e^{j\omega}) + S_{VV}(e^{j\omega}) \]

where

\[ S_{ZZ}(e^{j\omega}) = \frac{\sigma_w^2}{|A(e^{j\omega})|^2} + \sigma_v^2 \]
3. In this example we use the following z-transform pair:

\[
\alpha^{|k|} \leftrightarrow \frac{1 - \alpha^2}{(1 - \alpha z^{-1})(1 - \alpha z)} \quad (1)
\]

(a) The power spectrum of \( x(n) \) is

\[
S_{XX}(z) = \frac{3/4}{(1 - 1/2 z^{-1})(1 - 1/2 z)}
\]

the power spectrum of \( y(n) \) is given by

\[
S_{YY}(z) = H(z)H(1/z)S_{XX}(z) = \frac{3/4}{(1 - 1/3 z^{-1})(1 - 1/3 z)}
\]

(b) Using the z-transform pair mention above, we have

\[
\left( \frac{1}{3} \right)^{|k|} \leftrightarrow \frac{8/9}{(1 - 1/3 z^{-1})(1 - 1/3 z)}
\]

Therefore the autocorrelation of \( y(n) \) is given by

\[
R_{YY}(k) = \frac{27}{32} \left( \frac{1}{3} \right)^{|k|}
\]

(c) The cross correlation between \( x(n) \) and \( y(n) \) is given by

\[
R_{XY}(k) = R_{XX}(k) \ast h(-k)
\]

In z-transform notation the above relation can be expressed as follows:

\[
S_{XY}(z) = S_{XX}(z)H(1/z) = \frac{3/4}{(1 - 1/2 z^{-1})(1 - 1/2 z)} \times \frac{1}{(1 - 1/3 z)}
\]

\[
= \frac{3/4}{(1 - 1/2 z^{-1})(1 - 1/3 z)}
\]

Writing the above in terms of \( z^{-1} \) and then expanding by partial fraction expansion gives

\[
S_{XY}(z) = \frac{3}{4} \frac{z^{-1}}{(1 - 1/2 z^{-1})(z^{-1} - 1/2)} = \frac{9}{10} \frac{1}{1 - 1/2 z^{-1}} + \frac{3}{10} \frac{1}{z^{-1} - 1/3}
\]
Taking inverse z-transforms we have

\[ R_{XY}(k) = (9/10)(1/2)^k u(k) + (9/10)(3)^k u(-k - 1) \]

(d) The cross power density spectrum is calculated in (c) as \( S_{XY}(z) \).

4. (a) The power density spectrum is obtained by taking the DTFT of autocorrelations function:

\[
S_{XX}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_{XX}(k)e^{-jk\omega} \\
= \sum_{k=-\infty}^{\infty} \{2\delta(k) + j\delta(k - 1) - j\delta(k + 1)\}e^{-jk\omega} \\
= 2 + je^{-j\omega} - je^{j\omega} = 2 + 2\sin\omega
\]

(b) With \( \alpha \) real, using the DTFT pair:

\[
|\alpha| = \frac{1 - \alpha^2}{|1 - \alpha e^{-j\omega}|^2}
\]

we have

\[
S_{XX}(e^{j\omega}) = 1 + 2 \frac{1 - (1/4)}{|1 - (1/2)e^{-j\omega}|^2} = 1 + \frac{3/2}{(5/4) - \cos\omega} = \frac{11 - 4\cos\omega}{5 - 4\cos\omega}
\]

(c) Using the DTFT pair:

\[ e^{j\omega_0} \leftrightarrow 2\pi\delta(\omega - \omega_0) \]

the power density spectrum of

\[ R_{XX}(k) = 2\delta(k) + \cos(\pi k/4) = 2\delta(k) + \frac{1}{2}\left\{e^{j\pi k/4} + e^{-j\pi k/4}\right\} \]

is given by

\[
S_{XX}(e^{j\omega}) = 2 + \pi\delta(\omega - \frac{\pi}{4}) + \pi\delta(\omega + \frac{\pi}{4})
\]

5. (a) Expanding the power density spectrum in terms of complex exponentials we have

\[
S_{XX}(e^{j\omega}) = 3 + 2\cos\omega = 3 + e^{-j\omega} + e^{j\omega}
\]
Taking inverse DTFT we get the autocorrelation function:

\[ R_{XX}(k) = 3\delta(k) + \delta(k - 1) + \delta(k + 1) \]

(b) Using the DTFT pair:

\[ \alpha^{|k|} \leftrightarrow \frac{1 - \alpha^2}{(1 - \alpha e^{-j\omega})(1 - \alpha e^{j\omega})} = \frac{1 - \alpha^2}{(1 + \alpha^2) - 2\alpha \cos \omega} \]

it follows that

\[ \frac{4}{5 + 3\cos \omega} = \frac{8}{10 + 6\cos \omega} = \frac{8/9}{(10/9) + (6/9)\cos \omega} \]

This gives the autocorrelation function:

\[ R_{XX}(k) = (-\frac{1}{3})^{|k|} \]

(c)

Factorizing the power density spectrum we have

\[ S_{XX}(z) = \frac{-2z^2 + 5z - 2}{3z^2 + 10z + 3} = \frac{-2z + 5 - 2z^{-1}}{(3 + z)(3 + z^{-1})} = \frac{1}{9} \frac{(-2z + 5 - 2z^{-1})}{(1 + \frac{1}{3}z)(1 + \frac{1}{3}z^{-1})} \]

Using the z-transform pair (1) we have:

\[ (-\frac{1}{3})^{|k|} \leftrightarrow \frac{8/9}{(1 + (1/3)z)(1 + (1/3)z^{-1})} \]

Now using the above result we take the inverse Z-transform of \( S_{XX}(z) \) to yield:

\[ R_{XX}(k) = \frac{5}{8} (-\frac{1}{3})^{|k|} - \frac{2}{8} (-\frac{1}{3})^{|k-1|} - \frac{2}{8} (-\frac{1}{3})^{|k+1|} \]

6. (a)

\[ X_T(\Omega) = \int_{-T}^{T} x_T(t) e^{-j\Omega t} dt = \int_{-T}^{T} x(t) e^{-j\Omega t} dt \]

\[ X_T(-\Omega) = \int_{-T}^{T} x_T(t) e^{j\Omega t} dt = \int_{-T}^{T} x(t) e^{j\Omega t} dt = x_T^*(\Omega) \]
\[ S_{XX}(\Omega) = \lim_{T \to \infty} \frac{E\{X_T^*(\Omega)X_T(\Omega)\}}{2T} \]

\[ S_{XX}(-\Omega) = \lim_{T \to \infty} \frac{E\{X_T^*(-\Omega)X_T(-\Omega)\}}{2T} \]

\[ = \lim_{T \to \infty} \frac{E\{X_T(\Omega)X_T^*(\Omega)\}}{2T} = S_{XX}(\Omega) \]

(b)

\[ \dot{x}_T(t) = \begin{cases} \frac{dx(t)}{dt} & -T < t < T \\ 0 & \text{else where} \end{cases} \]

\[ = \begin{cases} \lim_{\epsilon \to 0} \frac{x(t+\epsilon)-x(t)}{\epsilon} & -T < t < T \\ 0 & \text{else where} \end{cases} \]

\[ \dot{X}_T(\Omega) = \mathcal{F}\{\dot{x}_T(t)\} = \lim_{\epsilon \to 0} \left[ \mathcal{F}\{x(t+\epsilon)-x(t)\} \right] \]

Using the Fourier transform pair:

\[ x(t-t_0) \leftrightarrow X(\Omega)e^{-j\Omega t_0} \]

\[ \dot{X}_T(\Omega) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ X_T(\Omega)e^{j\Omega \epsilon} - X_T(\Omega) \right] \]

\[ = X_T(\Omega) \lim_{\epsilon \to 0} e^{j\Omega \epsilon/2} \left( e^{j\Omega \epsilon/2} - e^{-j\Omega \epsilon/2} \right) \]

\[ = X_T(\Omega) \lim_{\epsilon \to 0} e^{j\Omega \epsilon/2} \frac{(2j\Omega)}{(2j\Omega \epsilon)} \]

\[ = X_T(\Omega) \lim_{\epsilon \to 0} e^{j\Omega \epsilon/2} \frac{\sin(\Omega \epsilon/2)}{(\Omega \epsilon/2)} \]

\[ = X_T(\Omega) j\Omega \]

where

\[ x_T(t) \leftrightarrow X_T(\Omega) \]
The power density spectrum of $\dot{X}(t)$ is given by

$$S_{\dot{X}\dot{X}}(\Omega) = \lim_{T \to \infty} \frac{E[\dot{X}_T^* (\Omega) \dot{X}_T (\Omega)]}{2T}$$

$$= \lim_{T \to \infty} \frac{E[-j\Omega X_T(\Omega) j\Omega X_T(\Omega)]}{2T}$$

$$= \Omega^2 E[|X_T(\Omega)|^2]/2T$$

7. (a) For real random processes $X(t)$ and $Y(t)$, we have

$$S_{XY}(\Omega) = \lim_{T \to \infty} \frac{E[X_T^*(\Omega) Y_T(\Omega)]}{2T}$$

$$S_{YX}(\Omega) = \lim_{T \to \infty} \frac{E[Y_T^*(\Omega) X_T(\Omega)]}{2T}$$

$$S_{YX}^*(-\Omega) = \lim_{T \to \infty} \frac{E[Y_T(-\Omega) X_T(-\Omega)]}{2T}$$

$$= \lim_{T \to \infty} \frac{E[X_T^*(\Omega) Y_T(\Omega)]}{2T} = S_{XY}(\Omega)$$

(b) and (c)

$$S_{XY}(\Omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\Omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cos(\Omega \tau) d\tau + (-j) \int_{-\infty}^{\infty} R_{XY}(\tau) \sin(\Omega \tau) d\tau$$

$$S_{XY}(-\Omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{j\Omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cos(\Omega \tau) d\tau + j \int_{-\infty}^{\infty} R_{XY}(\tau) \sin(\Omega \tau) d\tau$$

Comparing the two equations above we have

$$Re [S_{XY}(\Omega)] = Re [S_{XY}(-\Omega)]$$

$$Im [S_{XY}(\Omega)] = Im [-S_{XY}(-\Omega)]$$
8.

\[ R_{XX}(\tau) = Ae^{-a|\tau|}, \quad A \text{ and } a \text{ are positive constants} \]

\[ h(t) = e^{-bt}u(t), \quad b \text{ is a positive constant} \]

Using the following Fourier transform pairs

\[ e^{-at}u(t), a > 0 \leftrightarrow \frac{1}{j\Omega + a} \]
\[ e^{-a|t|}, a > 0 \leftrightarrow \frac{2a}{\Omega^2 + a^2} \]

\[ S_{XX}(\Omega) = \mathcal{F}\{R_{XX}(\tau)\} = A, \frac{2a}{\Omega^2 + a^2} \]

\[ H(\Omega) = \mathcal{F}\{h(t)\} = \frac{1}{j\Omega + b} \]

\[ |H(\Omega)|^2 = \frac{1}{j\Omega + b} \times \frac{1}{-j\Omega + b} = \frac{1}{\Omega^2 + b^2} \]

\[ S_{YY}(\Omega) = |H(\Omega)|^2 S_{XX}(\Omega) = \frac{1}{\Omega^2 + b^2} A, \frac{2a}{\Omega^2 + a^2} \]

\[ = \frac{aA}{a^2 - b^2} \left( \frac{2b}{\Omega^2 + b^2} \right) - \frac{A}{a^2 - b^2} \left( \frac{2a}{\Omega^2 + a^2} \right) \]

Taking inverse Fourier transforms of the above equation we get the autocorrelation function:

\[ R_{YY}(\tau) = \frac{aA}{a^2 - b^2} e^{-b|\tau|} - \frac{A}{a^2 - b^2} e^{-a|\tau|} \]

Reference: