1. Suppose that a signal $x(n)$ is to be modeled as the unit sample response of a second order all pole filter,

$$H(z) = \frac{b(0)}{1 + a(1)z^{-1} + a(2)z^{-2}}$$

The first 20 values of this signal are measured and found to be

$$x = [1, -1, 1, -1, \ldots, 1, -1]$$

Determine the coefficients, $a(1)$ and $a(2)$ using autocorrelation and covariance methods.

2. If $r_x(0) = 1$, $r_x(1) = 0.5$, and $r_x(2) = 0.75$, find the values of $a(1)$, $a(2)$, and $b(0)$ in the following AR(2) model for $x(n)$.

$$x(n) + a(1)x(n-1) + a(2)x(n-2) = b(0)w(n)$$

where $w(n)$ is unit variance white noise.

3. Use the method of spectral factorization to find a moving average model of order 2 for a process whose autocorrelation sequence is

$$r_x = [3, 1.5, 1]^T$$

4. Suppose that the first 5 values in the autocorrelation sequence for the process $x(n)$ are

$$r_x = [3, 9/4, 9/8, 9/16, 9/32, \ldots]^T$$

(a) Use the modified Yule-Walker equation method to find an ARMA(1,1) model for $x(n)$.

(b) Are the given values in the autocorrelation sequence consistent with the model that you found in part (a)?
5. We have a signal $x(n)$ for which we would like to obtain an all-pole model of the form

$$H(z) = \frac{b(0)}{1 + a(1)z^{-1} + a(2)z^{-2}}$$

Using the autocorrelation method, find explicit formulas for $b(0)$, $a(1)$, and $a(2)$ in terms of $r_x(0)$, $r_x(1)$, $r_x(2)$. 
1. **Autocorrelation Method**

To solve for the coefficients $a(k)$, Prony’s method is applied to the windowed sequence:

$$\tilde{x}(n) = \begin{cases} x(n) & n = 0, 1, 2, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

The normal equations are

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = - \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

where

$$r_x(k) = \sum_{n=0}^{\infty} \tilde{x}(n)\tilde{x}(n-k) = \sum_{n=k}^{N} x(n)x(n-k)$$

Evaluating, we have

$$r_x(0) = 20$$

$$r_x(1) = -19$$

$$r_x(2) = 18$$

Substituting these values into the normal equations, we have

$$\begin{bmatrix} 20 & -19 \\ -19 & 20 \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = - \begin{bmatrix} -19 \\ 18 \end{bmatrix}$$

Solving we have

$$a(1) = 0.9744$$

$$a(2) = 0.0256$$

Therefore, the denominator polynomial is

$$A(z) = 1 + 0.9744z^{-1} + 0.0256z^{-2}$$
The modelling error is given by

\[ E_p = r_x(0) + a(1)r_x(1) + a(2)r_x(2) = 1.9487 \]

Setting \( b(0) = \sqrt{E_p} \) to satisfy the energy matching constraint the model becomes

\[ H(z) = \frac{1.3960}{1 + 0.9744z^{-1} + 0.0256z^{-2}} \]

**Covariance Method**

The normal equations for finding the all-pole model parameters are:

\[
\begin{bmatrix}
  r_x(1, 1) & r_x(1, 2) \\
  r_x(2, 1) & r_x(2, 2)
\end{bmatrix}
\begin{bmatrix}
  a(1) \\
  a(2)
\end{bmatrix}
= -
\begin{bmatrix}
  r_x(1, 0) \\
  r_x(2, 0)
\end{bmatrix}
\]

where

\[ r_x(k, l) = \sum_{n=p}^{N} x(n-l)x(n-k) = \sum_{n=2}^{19} x(n-l)x(n-k) \]

Calculating the autocorrelations, we have

\[ r_x(1, 1) = r_x(2, 2) = r_x(2, 0) = 18 \quad r_x(1, 2) = r_x(2, 1) = r_x(1, 0) = -18 \]

Substituting these values into the normal equations, we have

\[
\begin{bmatrix}
  18 & -18 \\
  -18 & 18
\end{bmatrix}
\begin{bmatrix}
  a(1) \\
  a(2)
\end{bmatrix}
= 
\begin{bmatrix}
  -18 \\
  18
\end{bmatrix}
\]

Note that these equations are singular, indicating that a lower order model is possible. Therefore, calculating the first order AR model parameters, we have

\[ a(1) = \frac{r_x(1, 0)}{r_x(1, 1)} = 1 \]

Thus, setting \( b(0) = 1 \), we have

\[ H(z) = \frac{1}{1 + z^{-1}} \]

The model produces an exact fit to the data for \( n = 0, 1, 2, \ldots N \).
2. The Normal equation we want to solve is given by

\[ R_x a = -r_x \]

\[
\begin{bmatrix}
1.0 & 0.5 \\
0.5 & 1.0
\end{bmatrix}
\begin{bmatrix}
a(1) \\
a(2)
\end{bmatrix}
= -
\begin{bmatrix}
0.5 \\
0.75
\end{bmatrix}
\]

Solving the above equation, we have

\[
\begin{bmatrix}
a(1) \\
a(2)
\end{bmatrix}
= -
\frac{1}{0.75}
\begin{bmatrix}
1.0 & -0.5 \\
-0.5 & 1.0
\end{bmatrix}
\begin{bmatrix}
0.5 \\
0.75
\end{bmatrix}
= -
\frac{4}{3}
\begin{bmatrix}
0.125 \\
0.5
\end{bmatrix}
= -
\begin{bmatrix}
1/6 \\
2/3
\end{bmatrix}
\]

The minimum error is given by

\[ E_p = r_x(0) + a(1)r_x(1) + a(2)r_x(2) = 1 - \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) - \left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = \frac{5}{12} \]

Therefore

\[ b(0) = \sqrt{E_p} = \sqrt{\frac{5}{12}} \]

3. We are given the autocorrelation sequence:

\[ r_x = [3, 1.5, 1]^T \]

The power spectrum of \(x(n)\) is given by

\[ P_x(z) = \sum_{k=0}^{\infty} r_x(k)z^{-k} = 3 + 1.5(z^{-1} + z) + (z^{-2} + z^2) \]

which is a fourth order polynomial. Using spectral factorization, power spectrum can be factorized as follows:

\[ P_x(z) = \sigma_x^2 \left(1 + b(1)z^{-1} + b(2)z^{-2}\right) \left(1 + b(1)z + b(2)z^2\right) \]

Using either polynomial factoring program in matlab or by trial and error, we can obtain

\[ P_x(z) = \frac{1}{2} \left(1 + z^{-1} + 2z^{-2}\right) \left(1 + z + 2z^2\right) \]
Therefore the moving average model for $x(n)$ is given by

$$B(z) = \frac{1}{\sqrt{2}} \left(1 + z^{-1} + 2z^{-2}\right)$$

Note that in general, spectral factorization is not an easy problem to solve, if the order is high.

4. The Yule Walker equations for ARMA(1,1) model are

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \\ r_x(2) & r_x(1) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \end{bmatrix} = \begin{bmatrix} c_1(0) \\ c_1(1) \\ 0 \end{bmatrix}$$

The modified Yule Walker equations for determining the denominator coefficient is

$$r_x(1)a(1) = -r_x(2)$$

which gives $a(1) = -r_x(2)/r_x(1) = -1/2$. To determine MA coefficients, we compute $c_1(0)$ and $c_1(1)$ using Yule Walker equations as follows:

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \end{bmatrix} = \begin{bmatrix} c_1(0) \\ c_1(1) \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & 9/4 \\ 9/4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 15/8 \\ 3/4 \end{bmatrix}$$

and

$$[C_1(z)]_+ = \frac{15}{8} + \frac{3}{4}z^{-1}$$

Multiplying the above equation by $A_1^*(1/z^*) = 1 - (1/2)z$, we have

$$[C_1(z)]_+ A_1^*(1/z^*) = \left(\frac{15}{8} + \frac{3}{4}z^{-1}\right) \left(1 - (1/2)z\right) = -\frac{15}{16}z + \frac{3}{2} + \frac{3}{4}z^{-1}$$

Therefore the causal part of $P_y(z)$ is

$$[P_y(z)]_+ = [[C_1(z)]_+ A_1^*(1/z^*)]_+ = \frac{3}{2} + \frac{3}{4}z^{-1}$$
Using symmetry of $P_y(z)$

$$C_1(z)A_1^*(1/z^*) = \frac{3}{4}z + \frac{3}{2} + \frac{3}{4}z^{-1}$$

Spectral factorization of the above equation gives

$$P_y(z) = B(z)B^*(1/z^*) = \frac{3}{4}(1 + z^{-1})(1 + z)$$

The ARMA(1,1) model is given by

$$H(z) = \sqrt{3} \frac{1 + z^{-1}}{2(1 - \frac{1}{3}z^{-1})}$$

(b)

Yes. The model matches $r_x(k)$ for $k = 0, 1, 2, \ldots$ and for $k > 2$, note that

$$r_x(k) = \frac{1}{2}r_x(k - 1)$$

which they do.

5. The normal equation for autocorrelation method are

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = - \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

Solving for the denominator coefficients, we have

$$\begin{bmatrix} a(1) \\ a(2) \end{bmatrix} = -\frac{1}{r_x^2(0) - r_x^2(1)} \begin{bmatrix} r_x(0) & -r_x(1) \\ -r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

$$= -\frac{1}{r_x^2(0) - r_x^2(1)} \begin{bmatrix} r_x(0)r_x(1) - r_x(1)r_x(2) \\ -r_x^2(1) + r_x(0)r_x(2) \end{bmatrix}$$

and

$$b^2(0) = r_x(0) + a(1)r_x(1) + a(2)r_x(2)$$