1. In this problem we consider the design of a three-step predictor using a first order filter:

\[ W(z) = w(0) + w(1)z^{-1} \]

In other words, with \( x(n) \) the input to the predictor \( W(z) \), the output

\[ \hat{x}(n + 3) = w(0)x(n) + w(1)x(n - 1) \]

is the minimum mean-square estimate of \( x(n+3) \).

(a) What are the Wiener Hopf equations for the Wiener three-step predictor?

(b) If the values of \( r_x(k) \) for lags \( k = 0, 1, 2, 3, 4 \) are:

\[ r_x = \begin{bmatrix} 1.0, & 0, & 0.1, & -0.2, & -0.9 \end{bmatrix} \]

Solve the Wiener Hopf equations and find the optimum three-step predictor.

2. Let \( d(n) \) be an AR(1) process with an autocorrelation sequence:

\[ r_d(k) = \alpha^{|k|} \]

with \( 0 < \alpha < 1 \), and suppose that \( d(n) \) is observed in the presence of uncorrelated white noise, \( v(n) \) that has a variance of \( \sigma^2_v \),

\[ x(n) = d(n) + v(n) \]

Design a second-order Wiener filter and compare the mean-square error for the second-order filter with the mean-square error of the first order filter.

3. A random process \( x(n) \) is generated as follows:

\[ x(n) = \alpha x(n - 1) + v(n) + \beta v(n - 1) \]

where \( v(n) \) is a zero mean white noise with variance, \( \sigma^2_v \).
Design a first-order predictor

$$\hat{x}(n+1) = w(0)x(n) + w(1)x(n-1)$$

that minimizes the mean-square error in the prediction of $x(n+1)$ and find the minimum mean square error.

4. Consider the system shown in the figure below for estimating a process $d(n)$ from $x(n)$.

If $\sigma^2_d = 4$ and

$$r_x = \begin{bmatrix} 1.0, & 0.5, & 0.25 \end{bmatrix}^T$$

$$r_{dx} = \begin{bmatrix} -1.0, & 1.0 \end{bmatrix}^T$$

find the value of $a(1)$ that minimizes the mean-square error $J = E\{|e(n)|^2\}$, and find the mean-square error.

Reference:

Solutions

1. (a)

The Weiner Hopf equations are

\[ \mathbf{R}_x \mathbf{w} = \mathbf{r}_{dx} \]

where the elements of \( \mathbf{r}_{dx} \) are given by

\[ r_{dx}(k) = E \{ d(n)x(n - k) \} \]

For a 3 step predictor the desired signal, \( d(n) = x(n + 3) \). Therefore

\[ r_{dx}(k) = E \{ x(n + 3)x(n - k) \} = r_x(k + 3) \]

Since a first order Wiener Hopf filter

\[ W(z) = w(0) + w(1)z^{-1} \]

has 2 coefficients to be determined, the Weiner Hopf equation is

\[
\begin{bmatrix}
  r_x(0) & r_x(1) \\
  r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
  w(0) \\
  w(1)
\end{bmatrix}
= \begin{bmatrix}
  r_x(3) \\
  r_x(4)
\end{bmatrix}
\]

(b)

Substituting for the given \( r_x(k) \) in the above Wiener Hopf equation, we have

\[
\begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  w(0) \\
  w(1)
\end{bmatrix}
= \begin{bmatrix}
  -0.2 \\
  -0.9
\end{bmatrix}
\]

Solving we have

\[
\begin{bmatrix}
  w(0) \\
  w(1)
\end{bmatrix}
= \begin{bmatrix}
  -0.2 \\
  -0.9
\end{bmatrix}
\]

2. For a second order Wiener filter, the Wiener Hopf equation is

\[
\begin{bmatrix}
  r_x(0) & r_x(1) & r_x(2) \\
  r_x(1) & r_x(0) & r_x(1) \\
  r_x(2) & r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
  w(0) \\
  w(1) \\
  w(2)
\end{bmatrix}
= \begin{bmatrix}
  r_{dx}(0) \\
  r_{dx}(1) \\
  r_{dx}(2)
\end{bmatrix}
\]
Since \( x(n) \) and \( v(n) \) (with zero mean) are not correlated, we have

\[
r_{dx}(k) = r_{d}(k) = \alpha^{|k|} \quad \text{and} \quad r_{x}(k) = r_{d}(k) + r_{v}(k) = \alpha^{|k|} + \sigma_{v}^{2}\delta(k)
\]

Thus, the Wiener Hopf equation becomes

\[
\begin{bmatrix}
1 + \sigma_{v}^{2} & \alpha & \alpha^{2} \\
\alpha & 1 + \sigma_{v}^{2} & \alpha \\
\alpha^{2} & \alpha & 1 + \sigma_{v}^{2}
\end{bmatrix}
\begin{bmatrix}
w(0) \\
w(1) \\
w(2)
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
\alpha \\
\alpha^{2}
\end{bmatrix}
\]

With \( \alpha = 0.8 \) and \( \sigma_{v}^{2} = 1 \) we have

\[
\begin{bmatrix}
2 & 0.8 & 0.64 \\
0.8 & 2 & 0.8 \\
0.64 & 0.8 & 1
\end{bmatrix}
\begin{bmatrix}
w(0) \\
w(1) \\
w(2)
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0.8 \\
0.64
\end{bmatrix}
\]

Solving we have

\[
\begin{bmatrix}
w(0) \\
w(1) \\
w(2)
\end{bmatrix}
= 
\begin{bmatrix}
0.3824 \\
0.2 \\
0.1176
\end{bmatrix}
\]

Therefore the Wiener filter is given by

\[
W(z) = 0.3824 + 0.2z^{-1} + 0.1176z^{-2}
\]

The mean-square error is given by

\[
J_{\text{min}} = r_{d}(0) - w(0)r_{dx}(0) - w(1)r_{dx}(1) - w(2)r_{dx}(2) = 0.3824
\]

Recall from Notes-3 that the mean-square error for a first order Wiener filter, which was \( J_{\text{min}} = 0.4048 \). As expected, the second order filter produces smaller mean-square error.

3. It is given that the random process, \( x(n) \) is generated by the difference equation

\[
x(n) = \alpha x(n-1) + v(n) + \beta v(n-1)
\]

where, \( v(n) \) is a zero mean white noise with variance \( \sigma_{v}^{2} \). Taking z-transforms of the above equation and simplifying, we get

\[
\frac{X(z)}{V(z)} = \frac{1 + \beta z^{-1}}{1 - \alpha z^{-1}}
\]
In other words, the random process is generated by filtering \( v(n) \) by the following filter:

\[
H(z) = \frac{1 + \beta z^{-1}}{1 - \alpha z^{-1}}
\]

Therefore the power density spectrum of \( x(n) \) is given by

\[
P_x(z) = \frac{\sigma_v^2}{1 - \alpha^2} \left\{ \frac{1}{(1 - \alpha z^{-1})(1 - \alpha z)} \right\} \left\{ (1 + \beta^2) + \beta z^{-1} + \beta z \right\}
\]

Therefore, autocorrelation of \( x(n) \) is given by taking inverse z-transforms (from tables)

\[
r_x(k) = \frac{\sigma_v^2}{1 - \alpha^2} \left[ (1 + \beta^2)\alpha^{|k|} + \beta \alpha^{|k-1|} + \beta \alpha^{|k+1|} \right]
\]

Given the values of \( \alpha \), \( \beta \), and \( \sigma_v^2 \) we can solve the following Wiener Hopf equation

\[
\begin{bmatrix}
  r_x(0) & r_x(1) \\
  r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
w(0) \\
w(1)
\end{bmatrix}
= \begin{bmatrix}
r_x(1) \\
r_x(2)
\end{bmatrix}
\]

to determine the coefficients of the first order linear predictor:

\[
\hat{x}(n + 1) = w(0)x(n) + w(1)x(n - 1)
\]

The minimum mean-square error is given by

\[
J_{\text{min}} = r_x(0) - \sum_{k=0}^{p-1} w(k) r_x^*(k + 1) = r_x(0) - w(0)r_x(1) - w(1)r_x(2)
\]

4. The expression for error is given by

\[
e(n) = d(n) - \hat{d}(n) = d(n) - x(n) - a(1)x(n - 1)
\]

The mean-square error is given by

\[
J = E\{|e(n)|^2\} = E\{e(n)e^*(n)\}
= E\{[d(n) - x(n) - a(1)x(n - 1)][d(n) - x(n) - a(1)x(n - 1)]^*\}
\]
The value of coefficient $a(1)$ that minimizes the error $J$ is obtained by setting the derivative of $J$ with respect to $a^*(1)$ equal to zero and solving for $a(1)$ as shown below:

$$\frac{\partial J}{\partial a^*(1)} = -E\{e(n)x^*(n-1)\} = 0$$

Substituting for $e(n)$ in the above expression we have

$$E \{d(n)x^*(n-1) - x(n)x^*(n-1) - a(1)x(n-1)x^*(n-1)\} = 0$$

or

$$r_{dx}(1) - r_x(1) - a(1)r_x(0) = 0$$

Solving for $a(1)$ we have

$$a(1) = \frac{r_{dx}(1) - r_x(1)}{r_x(0)} = \frac{1}{2}$$

The minimum mean square error is given by

$$J = E\{e(n)[d(n) - x(n) - a(1)x(n-1)]^*\}$$

Using the orthogonality condition

$$E\{e(n)x^*(n-1)\} = 0$$

the minimum error $J$ becomes

$$J = E\{e(n)[d(n) - x(n)]^*\}$$

$$= r_d(0) - r_{dx}(0) - a(1)r_{dx}(1) - r_{dx}(0) + r_x(0) + a(1)r_x(1)$$

$$= 4 + 1 - 0.5 + 1 + 1 + 0.25$$

$$= 6.75$$