1. In this problem, we consider linear prediction in a noisy environment. Suppose that a signal \( d(n) \) is corrupted by noise,

\[
x(n) = d(n) + w(n)
\]

where \( r_w(k) = 0.5\delta(k) \) and \( r_{dw}(k) = 0 \). The signal \( d(n) \) is an AR(1) process that satisfies the difference equation

\[
d(n) = 0.5d(n-1) + v(n)
\]

where \( v(n) \) is white noise with variance \( \sigma_v^2 = 1 \). Assume that \( w(n) \) and \( v(n) \) are uncorrelated.

(a) Design a first-order FIR linear predictor

\[
W(z) = w(0) + w(1)z^{-1}
\]

for \( d(n) \), and find mean square prediction error

\[
J = E\{[d(n+1) - \hat{d}(n+1)]^2\}
\]

(b) Design a causal Wiener predictor and compare the mean-square error with that found in part (a).

2. In this problem, we consider the design of a causal IIR wiener filter for \( p \)-step prediction

\[
\hat{x}(n+p) = \sum_{k=0}^{\infty} h(k)x(n-k)
\]

(a) If \( x(n) \) is a real-valued random process with power spectral density

\[
P_x(z) = \sigma_0^2Q(z)Q(1/z)
\]

find the system function of the causal Wiener filter that minimizes the mean-square error

\[
J = E\{|x(n+p) - \hat{x}(n+p)|^2\}
\]
(b) If $x(n)$ is an AR(1) process with power spectrum

$$P_x(z) = \frac{1 - a^2}{(1 - az^{-1})(1 - az)}$$

find the causal p-step linear predictor and evaluate the mean-square error.

(c) If $x(n)$ is an MA(2) process that is generated by the difference equation

$$x(n) = 4v(n) - 2v(n - 1) + v(n - 2)$$

where $v(n)$ is zero mean unit variance white noise find the system function of the two step (p=2) predictor and evaluate the mean-square error.

3. We would like to estimate a process $d(n)$ from noisy observations:

$$x(n) = d(n) + v(n)$$

where $v(n)$ is white noise with variance $\sigma_v^2 = 1$ and $d(n)$ is a wide-sense stationary random process with the first four values of the autocorrelation sequence given by

$$r_d = [1.5, 0, 1.0, 0]^T$$

Assume that $d(n)$ and $v(n)$ are not correlated. Our goal is to design an FIR filter to reduce the noise in $d(n)$. Hardware constaints, however, limit the filter to only three nonzero coefficients in $W(z)$.

(a) Derive the optimal 3-multiplier causal filter

$$W(z) = w(0) + w(1)z^{-1} + w(2)z^{-2}$$

for estimating $d(n)$ and evaluate the mean-square error $E\{|d(n) - \hat{d}(n)|^2\}$.

(b) Repeat part (a) for noncasual FIR filter:

$$W(z) = w(-1)z + w(0) + w(1)z^{-1}$$

Reference:

1. (a) The Wiener Hopf equation for the FIR filter is

\[ R_x w = r_{dx} \]

Since the noise, \( w(n) \) is uncorrelated with the signal, \( d(n) \), the cross-correlation \( r_{dx}(k) \) is

\[ r_{dx}(k) = E \{d(n + 1)[d(n - k) + w(n - k)]\} = r_d(k + 1) \]

Therefore

\[ P_{dx}(z) = Z \{r_d(k + 1)\} = zP_d(z) \]

The autocorrelation of \( x(n) \) is given by

\[ r_x(k) = E\{x(n)x(n - k)\} = r_d(k) + r_w(k) \]

Since the power spectrum of \( d(n) \) is

\[ P_d(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.5z)} \]

the autocorrelation is given by

\[ r_d(k) = Z^{-1}\{P_d(k)\} = r_d(k) = \frac{4}{3} \left( \frac{1}{2} \right)^{|k|} \]

With

\[ r_w(k) = \frac{1}{2} \delta(k) \]

the Wiener Hopf equation for the second-order predictor is

\[
\begin{bmatrix}
  11/6 & 2/3 \\
  2/3 & 11/6 \\
\end{bmatrix}
\begin{bmatrix}
  w(0) \\
  w(1) \\
\end{bmatrix} =
\begin{bmatrix}
  2/3 \\
  1/3 \\
\end{bmatrix}
\]

Solving we have

\[
\begin{bmatrix}
  w(0) \\
  w(1) \\
\end{bmatrix} =
\begin{bmatrix}
  12/35 \\
  2/35 \\
\end{bmatrix}
\]
with a minimum mean-square error of

\[ J_{\text{min}} = r_d(0) - w(0)r_d(1) - w(1)r_d(2) = (4/3) - (12/35)(2/3) - (2/35)(1/3) \]
\[ = 38/35 = 1.086 \]

(b) For the causal Wiener Filter, the system function is

\[ H(z) = \frac{1}{\sigma_0^2 Q(z)} \left[ \frac{P_{dx}(z)}{Q(z^{-1})} \right]_+ \]

Since

\[ P_x(z) = P_d(z) + P_w(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.5z)} + \frac{1}{2} \]
\[ = \frac{(13/8) - (1/4)z^{-1} - (1/4)z}{(1 - 0.5z^{-1})(1 - 0.5z)} \]

Spectral factorizing we have

\[ P_x(z) = 1.5856 \frac{(1 - 0.1577z^{-1})(1 - 0.1577z)}{(1 - 0.5z^{-1})(1 - 0.5z)} \]

Therefore

\[ Q(z) = \frac{1 - 0.1577z^{-1}}{1 - 0.5z^{-1}} \]

with \( P_{dx}(z) = zP_d(z) \), it follows that

\[ H(z) = \frac{1}{1.5866 (1 - 0.1577z^{-1})} \left[ \frac{z}{(1 - 0.5z^{-1})(1 - 0.5z)} \times \frac{1 - 0.5z}{(1 - 0.1577z)} \right]_+ \]
\[ = \frac{1}{1.5866 (1 - 0.1577z^{-1})} \left[ \frac{z}{(1 - 0.5z^{-1})(1 - 0.1577z)} \right]_+ \]
\[ = 0.6307 \frac{1 - 0.5z^{-1}}{(1 - 0.1577z^{-1})} \left[ \frac{1}{(1 - 0.5z^{-1})(z^{-1} - 0.1577)} \right]_+ \]

Expanding by partial fractions we have

\[ \frac{1}{(1 - 0.5z^{-1})(z^{-1} - 0.1577)} = \frac{0.5428}{1 - 0.5z^{-1}} + \frac{1.0856}{z^{-1} - 0.1577} \]
The causal part is

\[
\left[ \frac{z}{(1-0.5z^{-1})(1-0.1577z^{-1})} \right]_+ = \frac{0.5428}{1-0.5z^{-1}}
\]

Therefore

\[
H(z) = 0.6307 \frac{1-0.5z^{-1}}{1-0.1577z^{-1}} \times \frac{0.5428}{1-0.5z^{-1}} = \frac{0.3423}{1-0.1577z^{-1}}
\]

Thus the unit impulse response obtained by taking inverse z-transforms of the above relation is given by

\[
h(n) = 0.3423(0.1577)^n u(n)
\]

For the minimum mean-square error, we have

\[
J_{\text{min}} = r_d(0) - \sum_{l=0}^{\infty} h(l) r_{dx}(l) = \frac{4}{3} - \sum_{l=0}^{\infty} 0.3423(0.1577)^l \times \frac{4}{3} \left( \frac{1}{2} \right)^{l+1}
\]

\[
= \frac{4}{3} - 0.2282 \sum_{l=0}^{\infty} (0.07885)^l = 1.0856
\]

2. (a) The optimum causal Wiener filter is

\[
H(z) = \frac{1}{\sigma_0^2 Q(z)} \left[ \frac{P_{dx}(z)}{Q(z^{-1})} \right]_+
\]

with \( d(n) = x(n+p) \),

\[
r_{dx}(k) = E\{d(n)x(n-k)\} = E\{x(n+p)x(n-k)\} = r_x(k+p)
\]

\[
P_{dx}(z) = z^p P_x(z) = z^p \sigma_0^2 Q(z) Q(z^{-1})
\]

Therefore

\[
H(z) = \frac{1}{Q(z)} [z^p Q(z)]_+
\]

(b) With the power spectrum of \( x(n) \) given by

\[
P_x(z) = \frac{1 - \alpha^2}{(1 - \alpha z^{-1})(1 - \alpha z)}
\]
we see that $\sigma_0^2 = 1 - \alpha^2$, and

$$Q(z) = \frac{1}{1 - az^{-1}}$$

Therefore

$$[z^pQ(z)]_+ = \left[\frac{z^p}{1 - az^{-1}}\right]_+ = \left[z^p(1 + az^{-1} + a^2z^{-2} + \ldots)\right]_+$$

$$= a^p + a^{p+1}z^{-1} + a^{p+2}z^{-2} + \ldots = \frac{a^p}{1 - az^{-1}}$$

and the optimum causal Wiener filter for estimating $x(n + p)$ is

$$H(z) = \frac{1}{Q(z)} \left[z^pQ(z)\right]_+ = (1 - az^{-1}) \times \frac{a^p}{1 - az^{-1}} = a^p$$

Thus, the estimate of $x(n + p)$ is given by

$$\hat{x}(n + p) = a^p x(n)$$

Note that the predictor only uses the most recent value of the sequence, $x(n)$ to predict $x(n + p)$. Since $x(n)$ is an AR process, this value carries all of the information about the past history of $x(n)$. The minimum error is given by

$$J_{\text{min}} = r_d(0) - \sum_{l=0}^{\infty} h(l)r_{dx}^*(l)$$

With $r_d(k) = r_x(k) = a^{|k|}$, $r_{dx}(k) = r_x(k + p)$, and $h(n) = a^p \delta(n)$, this becomes

$$J_{\text{min}} = 1 - a^{2p}$$

(c) If $x(n)$ is a moving average process that is generated by the difference equation

$$x(n) = 4v(n) - 2v(n - 1) + v(n - 2)$$

then the power spectrum of $x(n)$ is

$$P_x(z) = (4 - 2z^{-1} + z^{-2})(4 - 2z + z^2) = 16(1 - 0.5z^{-1} + 0.25z^{-2})(1 - 0.5z + 0.25z^2)$$
With

\[ [z^pQ(z)]_+ = [z^p(1 - 0.5z^{-1} + 0.25z^{-2})]_+ \]

if \( p = 2 \), then this becomes

\[ [z^2Q(z)]_+ = \frac{1}{4} \]

Therefore, the optimum predictor is

\[
H(z) = \frac{1}{Q(z)} [z^2Q(z)]_+ = \frac{0.25}{1 - 0.5z^{-1} + 0.25z^{-2}}
\]

or

\[
\hat{x}(n+1) = 0.5\hat{x}(n) - 0.25\hat{x}(n-1) + 0.25x(n)
\]

For the minimum error, we have

\[
J_{min} = r_d(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})P_{dx}^*(e^{j\omega})d\omega
\]

With \( r_d(0) = r_x(0) = 21 \) and

\[
P_{dx}^*(e^{j\omega}) = e^{-jp\omega}P_x(e^{j\omega}) = e^{-jp\omega}16|1 - 0.5e^{-j\omega} + 0.25e^{-j2\omega}|^2
\]

we have with \( p = 2 \),

\[
J_{min} = 21 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{0.25}{1 - 0.5e^{-j\omega} + 0.25e^{-j2\omega}} \times 16e^{-j2\omega}|1 - 0.5e^{-j\omega} + 0.25e^{-j2\omega}|^2 d\omega
\]

Therefore,

\[
J_{min} = 21 - \frac{1}{2\pi} \int_{-\pi}^{\pi} 4e^{-j2\omega}[1 - 0.5e^{j\omega} + 0.25e^{j2\omega}]d\omega = 20
\]

3. (a) The Wiener Hopf equations for the optimal three multiplier filter

\[
W(z) = w(0) + w(1)z^{-1} + w(2)z^{-2}
\]

for estimating \( d(n) \) is

\[
\begin{bmatrix}
  r_x(0) & r_x(1) & r_x(2) \\
  r_x(1) & r_x(0) & r_x(1) \\
  r_x(2) & r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
  w(0) \\
  w(1) \\
  w(2)
\end{bmatrix}
= \begin{bmatrix}
  r_{dx}(0) \\
  r_{dx}(1) \\
  r_{dx}(2)
\end{bmatrix}
\]
With
\[ r_x(k) = r_d(k) + r_v(k) \]
and
\[ r_{dx}(k) = r_d(k) \]
using the given values for the autocorrelation, \( r_d(k) \), these equations become
\[
\begin{bmatrix}
2.5 & 0 & 1.0 \\
0 & 2.5 & 0 \\
1.0 & 0 & 2.5
\end{bmatrix}
\begin{bmatrix}
w(0) \\
w(1) \\
w(2)
\end{bmatrix}
= \begin{bmatrix} 1.5 \\ 0 \\ 1.0 \end{bmatrix}
\]
Solving we have
\[
\begin{bmatrix}
w(0) \\
w(1) \\
w(2)
\end{bmatrix}
= \begin{bmatrix} 0.5238 \\ 0 \\ 0.1905 \end{bmatrix}
\]
The mean square error is given by
\[
E\{|d(n) - \hat{d}(n)|^2\} = r_d(0) - \sum_{k=0}^{2} w(k)r_{dx}(k) = 1.5 - 0.5238 \times 1.5 - 0.1905 = 0.5238
\]
(b) For the noncausal FIR filter
\[ W(z) = w(-1)z + w(0) + w(1)z^{-1} \]
the Wiener-Hopf equations for the optimum coefficients may be derived as follows. With \( J \) the mean square error
\[
J = E\{e^2(n)\}
\]
we set the derivative of \( J \) with respect to \( w(k) \) equal to zero as follows:
\[
\frac{\partial J}{\partial w(k)} = E\{-2e(n)x(n - k)\} = 0, k = 0, \pm 1
\]
Substituting for \( e(n) \) and simplifying we have
\[
\sum_{l=-1}^{1} w(l)r_x(k - l) = r_{dx}(k), k = 0, \pm 1
\]
Since \( r_{dx}(k) = r_d(k) \), then these equations are

\[
\begin{bmatrix}
  r_x(0) & r_x(1) & r_x(2) \\
  r_x(1) & r_x(0) & r_x(1) \\
  r_x(2) & r_x(1) & r_x(0)
\end{bmatrix}
\begin{bmatrix}
  w(-1) \\
  w(0) \\
  w(1)
\end{bmatrix}
= 
\begin{bmatrix}
  r_{dx}(-1) \\
  r_{dx}(0) \\
  r_{dx}(1)
\end{bmatrix}
\]

Using the given autocorrelations, these become

\[
\begin{bmatrix}
  2.5 & 0 & 1.0 \\
  0 & 2.5 & 0 \\
  1.0 & 0 & 2.5
\end{bmatrix}
\begin{bmatrix}
  w(-1) \\
  w(0) \\
  w(1)
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  1.5 \\
  0
\end{bmatrix}
\]

Solving we have

\[
\begin{bmatrix}
  w(-1) \\
  w(0) \\
  w(1)
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0.6 \\
  0
\end{bmatrix}
\]

The mean square error is given by

\[
E\{|d(n) - \hat{d}(n)|^2\} = r_d(0) - \sum_{k=-1}^{1} w(k)r_{dx}(k) = 0.6
\]

Since \( r_d(1) = 0 \) and the additive noise, \( v(n) \) is white then \( x(n \pm 1) \) is of no use in estimating \( d(n) \). Therefore a better estimator to use that has only three coefficients is the following:

\[
W(z) = w(0) + w(2)z^{-2} + w(-2)z^2
\]