1. Find all solutions to $1 + 5 \times 2^m = n^2$, where $m, n$ are positive integers.

**Solution.** We start by rearranging:

$$1 + 5 \cdot 2^m = n^2$$
$$5 \cdot 2^m = n^2 - 1$$
$$= (n - 1)(n + 1).$$

Now, $n - 1$ and $n + 1$ have the same parity, and since $m$ is positive, we have $2 \mid (n - 1)(n + 1)$. Hence, in fact both $n - 1$ and $n + 1$ are even. Thus let $n - 1 = 2a$ and $n + 1 = 2a + 2$. Then

$$5 \cdot 2^m = 2a(2a + 2)$$
$$5 \cdot 2^{m-2} = a(a + 1).$$

Now one of $a$ and $a + 1$ is odd and the other even. Thus we have two cases.
Case 1: $a = 5$ and $2^{m-2} = a + 1$. Then $2^{m-2} = 5 + 1 = 6$ which is impossible.
Case 2: $a + 1 = 5$ and $2^{m-2} = a$. Then $a = 4 = 2^{m-2}$, whence $m = 4$ and $n = 2a + 1 = 9$.
Thus there is exactly one solution, namely $(m, n) = (4, 9)$ for which $n^2 = 81 = 1 + 5 \cdot 2^4$.

2. Let $f$ be a function taking positive integers to positive integers such that

(i) $f(ab) = f(a)f(b)$,
(ii) $f(a) < f(b)$ whenever $a < b$,
(iii) $f(3) \geq 7$.

Find the smallest value that $f(3)$ can take.

**Solution.** Observe that $f(x) = x^2$ satisfies the criteria. Assume this definition for $f$. Then

$$f(ab) = (ab)^2$$
$$= a^2 \cdot b^2 = f(a)f(b).$$

So (i) is satisfied. Condition (ii) says that $f$ should be strictly increasing, which is satisfied by $f(x) = x^2$ for $x$ positive, and in particular for $x \in \mathbb{N}$. Also,

$$f(3) = 3^2 = 9 > 7.$$ 

So (iii) is satisfied. So, we see that $f(3) = 9$ is possible.
We will be done, if we can show that $f(3) \not< 9$. 

We are given $f(a)$ is positive for all $a \in \mathbb{N}$. So, by (i),
\[ f(a) = f(a \cdot 1) = f(a)f(1) \]
\[ \therefore f(1) = \frac{f(a)}{f(a)} = 1, \text{ since } f(a) \neq 0. \]

So, now (ii) implies $f(2) > f(1)$, i.e. $f(2) \geq 2$.
Suppose $f(2) = 2$. Then, by (i), $f(4) = f(2 \cdot 2) = f(2)^2 = 4$. But then $f(3) < f(4) = 4$
violates (iii), a contradiction.
Suppose $f(2) = 3$. Then by (i), (ii) and (iii),
\[ f(27) = f(3 \cdot 3 \cdot 3) = f(3)^3 \geq 27^3 = 343 \]
\[ > 243 = 3^5 = f(2)^5 = f(2^5) = f(32), \]
i.e. we have $f(27) > f(32)$ violating (ii).
Thus $f(2) \geq 4$, so that $f(8) = f(2^3) = f(2)^3 \geq 64$, by (i). So by (i) and (ii),
\[ f(3)^2 = f(3^2) = f(9) > f(8) \geq 64 = 8^2 \]
\[ \therefore f(3) > 8, \]
i.e. $f(3) \not< 9$.
So the least value $f(3)$ can be is 9.

3. Let $PRUS$ be a trapezium with $PR \parallel SU$ such that $\angle SPU = 2 \angle UPR$ and $\angle PSR = 2 \angle RSU$. Suppose $Q$ is on $PR$ such that $QS$ bisects $\angle PSR$ and, similarly, $T$ is on $SU$ such that $TP$ bisects $\angle SPU$. Let $PT$ meet $SQ$ at $E$, and let $PU$ meet $SR$ at $F$. The line through $E$ parallel to $SR$ meets $PU$ at $G$, and the line through $E$ parallel to $PU$ meets $SR$ at $H$. Finally, the line through $G$ and $H$ meets $PR$ at $K$ and $SU$ at $L$.

Prove that $KG = GH = HL$.

4. Let $P_1, P_2, \ldots, P_n$ be $n$ different points on a circle. Between each pair of points there is a line segment which is coloured either red or blue. Consider colourings for which $P_iP_j$ is red if and only if $P_{i+1}P_{j+1}$ is blue, for any distinct $i$ and $j$ in the set $\{1, \ldots, n\}$. We interpret $P_{n+1}$ as being the same point as $P_1$.

a. For which values of $n$ is such a colouring possible?

b. Let a step consist of moving along a single red segment from one point to another point.

Show that it is possible to get from each point to any other point in at most three steps.

5. Let $ABCD$ be a square, and let $E$ be a point on its diagonal $BD$. Suppose that $O_1$ is the centre of the circle passing through $\triangle ABE$ and $O_2$ is the centre of the circle passing through $\triangle ADE$.

Show that $AO_1EO_2$ is a square.

6. For each positive integer $n$, let $a(n)$ denote the product of all digits of $n$.

a. Show that $a(n) \leq n$.

b. Find all solutions to the equation $n^2 - 17n + 56 = a(n)$.
7. For each sequence $S = (a_1, a_2, \ldots, a_n)$ of non-negative integers let the \textit{offspring} of $S$ be the sequence $T = (b_1, b_2, \ldots, b_n)$, where $b_i$ is the number of integers in $S$ to the right of $a_i$ that are less than $a_i$. For example,

if $S = (6, 1, 8, 0, 5, 7, 2, 2, 4, 0, 7, 7, 5)$,
then $T = (8, 2, 10, 0, 4, 5, 1, 1, 0, 1, 1, 0)$.

For a given sequence $S_0$, let $S_1$ be the offspring of $S_0$, $S_2$ be the offspring of $S_1$, and so on. Show that there exists an integer $j$ such that $S_j = S_{j+1}$.

8. Alice and Bob play the following guessing game. Alice chooses a positive integer $a$ and tells Bob that it is at most 2006. At each turn, Bob chooses a positive integer $b$ and calls it out to Alice. Alice then tells Bob whether or not $a + b$ is prime.

Prove that there is a strategy for Bob to determine $a$ in less than 2006 turns.