Australian Mathematics Olympiad, 2007 Problems

1. In $\triangle ABC$ let $AB$ be the shortest side. Let the midpoints of $BC$ and $AC$ be $X$ and $Y$, respectively. Suppose $P$ is on $AC$ such that $PX$ is perpendicular to $BC$. The circle passing through $A$, $B$ and $P$ meets the side $BC$ again at $Q$.

Prove that $QY$ is perpendicular to $AC$.

2. Let $p$ be an odd prime. Determine all pairs $(m, n)$ of positive integers that satisfy

$$(p - 1)(p^n + 1) = 4m(m + 1).$$

3. Consider an $m \times n$ rectangle divided into $mn$ unit squares using $m - 1$ horizontal grid lines and $n - 1$ vertical grid lines. The rectangle is to be cut into $mn$ unit squares in stages as follows.

The first stage is to cut the rectangle into two pieces along a grid line. Each subsequent stage consists of choosing one or more pieces and cutting each of these into two along one of its grid lines.

Find, in terms of $m$ and $n$, the minimum number of stages required.

4. a. Prove that for each positive integer $n$ there exists a real number $C_n$ such that

$$r + r^2 + r^3 + \cdots + r^{2n} \leq C_n(1 + r^{2n+1})$$

for all positive real numbers $r$.

b. For each $n$, find the minimum value of $C_n$.

5. Consider a $3 \times 3$ square array of distinct positive integers. It is called special if the eight products of the numbers in each row, each column and each diagonal are equal. For a special array let the common value of the products be $P$.

a. Show that there is exactly one such $P$ between 5000 and 6500.

b. Give an example of a special array with this value of $P$.

6. Let $a$ and $b$ be non-zero real numbers such that $a^2 + b^2 = 1$.

Prove that $|a + \frac{a}{b} + b + \frac{b}{a}| \geq 2 - \sqrt{2}$.

7. Let $ABC$ be an acute angled triangle. For each point $M$ on the segment $AC$, let $S_1$ be the circle through $A$, $B$ and $M$, and let $S_2$ be the circle through $M$, $B$ and $C$. Let $D_1$ be the disk bounded by $S_1$, and let $D_2$ be the disk bounded by $S_2$.

Show that the area of the intersection of $D_1$ and $D_2$ is smallest when $BM \perp AC$.

8. Let $S$ be a finite set of non-negative real numbers including 0. Suppose that whenever $a$ and $b$ belong to $S$, then at least one of $a + b$ and $|a - b|$ belongs to $S$.

Prove that

(i) $S$ consists of exactly four distinct real numbers

or

(ii) $S = \{0, r, 2r, \ldots, nr\}$ for some non-negative integer $n$ and some positive real number $r$. 