1. In $K$ be a circle with $PQ$ as diameter. Let $C$ be a circle with centre on $K$ and with $PQ$ tangent to $C$.

Prove that the other tangents from $P$ and $Q$ are parallel.

Solution. Let $O$ be the centre of $C$, let the other tangent from $P$ to $C$ touch $C$ at $R$, let the other tangent from $Q$ to $C$ touch $C$ at $S$, and let $PQ$ touch $C$ at $T$.

\[ \angle OTQ = \angle OSQ = 90^\circ, \quad (QT \text{ and } ST \text{ are tangents to } C) \]
\[ \text{OQ is common} \]
\[ OT = OS, \quad \text{(radii of } C) \]
\[ \therefore \triangle OTQ \cong \triangle OSQ, \quad \text{by the RHS Rule} \]
\[ \therefore \angle TOQ = \angle SOQ \]

Similarly,
\[ \triangle OTP \cong \triangle ORP \]
\[ \therefore \angle TOP = \angle ROP \]
\[ \angle POQ = 90^\circ, \quad \text{since } PQ \text{ is a diameter of } K \text{ and } O \text{ is on } K \]
\[ \therefore \angle ROS = \angle ROP + \angle TOP + \angle TOQ + \angle SOQ \]
\[ = 2(\angle TOP + \angle TOQ) \]
\[ = 2\angle POQ = 180^\circ \]
\[ \therefore RS \text{ is a straight line with } PR \perp RS \perp QS \]
\[ \therefore PR \parallel QS \]

Hence the tangents other than $PQ$ to $C$ from points $P$ and $Q$ are parallel.

2. Let $f(x) = 5^x$. Determine all real solutions of the equation

\[ f(x + f(2008)) = 2008 - x. \]

Solution. Hint. $g(x) = f(x + f(2008)) = 5^{x+c}$ where $c = 5^{2008}$ (constant) is a strictly increasing function of $x$; $h(x) = 2008 - x$ is a strictly decreasing function of $x$; there exists $x_1$ and $x_2$ such that $h(x_1) > g(x_1)$ and $g(x_2) > h(x_2)$. Putting this together gives the existence of a unique solution, or give a contradiction argument.

Then observe that $x = 2008 - f(2008) = 2008 - 5^{2008}$ satisfies the equation.

3. A positive integer is called square-free if it has no factor greater than 1 which is a perfect square.
For each positive integer \( n \), let \( f(n) \) be the sum of all square-free factors of \( n \). Determine all values of \( n \), for which \( f(n)/n \) is an integer.

**Solution.** Hint. Try small values of \( n \). Write \( n \) in terms of its prime decomposition. Deduce a formula for \( f(n) \) and hence a very restricting condition on the prime divisors of \( f(n)/n \) that shows that there are only a small finite number of solutions.

4. Find all positive integers \( n \) and all prime numbers \( p \) such that the polynomial

\[
x^5 + x + p^n
\]

can be written as the product of two polynomials with integer coefficients and positive degrees.

**Solution.** Let \( q(x) = x^5 + x + p^n = a(x)b(x) \) with \( \partial a \leq \partial b \) (where \( \partial u \) is the degree of the polynomial \( u(x) \)). The either \( \partial a = 1 \) and \( \partial b = 4 \) or \( \partial a = 2 \) and \( \partial b = 3 \).

Case 1: \( \partial a = 2 \) and \( \partial b = 3 \). Then, for some \( \alpha, \beta, \gamma, \delta, \varepsilon \),

\[
q(x) = (x^2 + \alpha x + \beta)(x^3 + \gamma x^2 + \delta x + \varepsilon)
\]

\[
= x^5 + \gamma x^4 + \delta x^3 + \varepsilon x^2 + \alpha x^4 + \alpha \gamma x^3 + \alpha \delta x^2 + \alpha \varepsilon x
\]

\[
+ \beta x^3 + \beta \gamma x^2 + \beta \delta x + \beta \varepsilon
\]

\[
= x^5 + (\gamma + \alpha) x^4 + (\delta + \alpha \gamma + \beta) x^3 + (\varepsilon + \alpha \delta + \beta \gamma) x^2 + (\alpha \varepsilon + \beta \delta)x + \beta \varepsilon
\]

\[
= x^5 + 0 \cdot x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + p^n
\]

Equating coefficients we have

\[
\gamma + \alpha = 0 \quad \Rightarrow \quad \gamma = -\alpha
\]

\[
\delta + \alpha \gamma + \beta = 0 \quad \Rightarrow \quad \delta = -\alpha \gamma - \beta = \alpha^2 - \beta
\]

\[
\varepsilon + \alpha \delta + \beta \gamma = 0 \quad \Rightarrow \quad \varepsilon = -\alpha \delta - \beta \gamma = \alpha(2\beta - \alpha^2)
\]

\[
\alpha \varepsilon + \beta \delta = 1 \quad \Rightarrow \quad \alpha \varepsilon + \beta \delta = 1
\]

\[
\beta \varepsilon = p^n \quad \Rightarrow \quad \alpha \beta(2\beta - \alpha^2) = p^n
\]

By (2) each of \( \alpha \) and \( \beta \) divide \( p^n \), and so each has magnitude that is a power of \( p \). However, \( p \) cannot divide both of \( \alpha \) and \( \beta \), since otherwise \( p \) would divide the righthand side of (1). So, one of \( \alpha \) and \( \beta \) is \( \pm 1 \).

Subcase (i): \( \alpha = \pm 1 \). Then by (1),

\[
3\beta - 1 - \beta^2 = 1
\]

\[
\beta^2 - 3\beta + 2 = 0
\]

\[
(\beta - 2)(\beta - 1) = 0
\]

so that \( \beta = 1 \) or 2; substitution in (2), give \( p^n = \pm 1 \) or \( \pm 6 \), respectively. Either way, \( p^n \) is not a power of a prime (contradiction).
Subcase (ii): $\beta = \pm 1$. Then by (1),

\[
\pm 3\alpha^2 - \alpha^4 - 1 = 1 \\
\alpha^4 - 3\alpha^2 + 2 = 0 \\
(\alpha^2 \mp 2)(\alpha^2 \mp 1) = 0
\]

so that $\alpha^2 = \pm 1$ or $\pm 2$. Since $\alpha$ is an integer, this gives $\alpha = \pm 1$, only, which was the premise of Subcase (i). So again we have a contradiction.

So we cannot have $\partial a = 2$ and \partial b = 3.

Case 2: $\partial a = 1$ and $\partial b = 4$. Let $a(x) = x - \kappa$. Then since $a(x)$ is a linear factor of $q(x)$, by the Factor Theorem,

\[
q(\kappa) = \kappa^5 + \kappa + p^n = 0 \\
\kappa(\kappa^4 + 1) = -p^n. 
\]

But $\gcd(\kappa, \kappa^4 + 1) = 1$, whereas each factor of the lefthand side of (3), namely $\kappa$ and $\kappa^4 + 1$, has absolute value necessarily a power of $p$. Thus $\kappa$ or $\kappa^4 + 1$ is $\pm 1$.

Subcase (i): $\kappa^4 + 1 = \pm 1$. Since $\kappa^4 + 1$ cannot be negative we have $\kappa^4 + 1 = 1$ which implies $\kappa = 0$ and consequently $-p^n = 0$ (contradiction).

Subcase (ii): $\kappa = \pm 1$. Since $\kappa^4 + 1$ is positive and $\kappa(\kappa^4 + 1) = -p^n < 0$, we have $\kappa = -1$ which implies $\kappa^4 + 1 = 2$ and consequently $-p^n = 2$, i.e.

\[
p = 2, n = 1.
\]

Indeed, with $n = 1$ and $p = 2$,

\[
x^5 + x + 2 = (x + 1)(x^4 - x^3 + x^2 - x + 2).
\]

Thus there is exactly one solution $(n, p) = (1, 2)$ for which

\[
x^5 + x + p^n
\]

can be written as the product of two positive-degree polynomials over the integers.

5. For each positive integer $m$, let $F(m)$ be the largest integer such that $10^{F(m)}$ divides $m!$.

Prove that there exists a positive integer $n$ such that for each $m$

\[
either F(m) \leq n \quad \text{or} \quad F(m) \geq n + 2008.
\]

Solution. Hint. In general, $N! = N \cdot (N - 1)!$. Use this to deduce an expression for $F(10^{2008})$ and observe that $F$ is monotonic increasing.

6. Let $ABCD$ be a convex quadrilateral. Suppose there is a point $P$ on the segment $AB$ with $\angle APD = \angle BPC = 45^\circ$.

If $Q$ is the intersection of the line $AB$ with the perpendicular bisector of $CD$, prove $\angle CQD = 90^\circ$. 

3
Solution. Let $R$ be the midpoint of $CD$. Thus $RQ$ is the perpendicular bisector of $CD$.

\[
\angle DPC = 180^\circ - \angle APD - \angle BPC
\]

\[
= 90^\circ
\]

\[\therefore P \text{ lies on the circle with diameter } CD\]
Call this circle $K$. Then

$D, P, C$ lie on $K$ which has centre $R$ and radius $RC$.

Let $PB$ intersect $K$ at $Q'$. Then

\[
\angle Q'PC = \angle BPC = 45^\circ
\]

and

\[
\angle Q'RC = 2\angle Q'PC = 90^\circ, \quad \text{angles at circumference and centre standing on common chord } Q'C
\]

\[\therefore Q' \text{ lies on perpendicular bisector of } CD \text{ and on } AB.\]
Thus $Q'$ is the intersection of the line $AB$ with the perpendicular bisector of $CD$.

\[\therefore Q' = Q \text{ lies on } K\]

\[\therefore \angle CQD = 90^\circ, \quad \text{(angle in a semicircle)}.\]

Alternative Method. Let $\theta = \angle RPC$. Then

\[
\angle RCP = \theta, \quad \text{since } RP = RC \text{ (radii of } K),
\]

so that $\triangle PRC$ is isosceles

\[\therefore \angle DRP = 2\theta, \quad \text{(sum of interior opposite angles of } \triangle PRC)\]

\[\therefore \angle PRQ = 180^\circ - \angle DRP - \angle CRQ
\]

\[= 90^\circ - 2\theta\]

\[\therefore \angle RQP = 180^\circ - \angle PRQ - \angle RPQ
\]

\[= 180^\circ - (90^\circ - 2\theta) - (45^\circ + \theta)
\]

\[= 45^\circ + \theta = \angle RPQ
\]

\[\therefore \triangle PRQ \text{ is isosceles}
\]

\[\therefore RP = RQ
\]

\[\therefore Q \text{ lies on } K, \text{ since } RP = RQ \text{ is a radius of } K
\]

\[\therefore \angle CQP = 90^\circ
\]

since $\angle CQP$ is an angle in a semicircle, as before.

7. Let $A_1A_2A_3$ and $B_1B_2B_3$ be triangles. If

\[p = A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1, \text{ and}
\]

\[q = A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 + A_3B_1 + A_3B_2 + A_3B_3,
\]

prove that $3p \leq 4q$.

Solution. For convenience, let $A_4 = A_1$ and $B_4 = B_1$, so that we may write $p$ and $q$ in $\Sigma$ notation as follows

\[p = \sum_{i=1}^{3} (A_iA_{i+1} + B_iB_{i+1}) \quad \text{and} \quad q = \sum_{i=1}^{3} \sum_{j=1}^{3} A_iB_j.
\]
Now, by the Triangle Inequality, we have for each \( i \) and each \( j \),
\[
A_iA_{i+1} \leq A_iB_j + B_jA_{i+1} \quad \text{and} \quad B_iB_{i+1} \leq B_iA_j + A_jB_{i+1},
\]
Thus we have
\[
3A_iA_{i+1} \leq \sum_{j=1}^{3} (A_iB_j + B_jA_{i+1}) \quad \text{and} \quad 3B_iB_{i+1} \leq \sum_{j=1}^{3} (B_iA_j + A_jB_{i+1})
\]
\[
3p = \sum_{i=1}^{3} 3(A_iA_{i+1} + B_iB_{i+1}) \leq \sum_{i=1}^{3} \sum_{j=1}^{3} (A_iB_j + B_jA_{i+1} + B_iA_j + A_jB_{i+1})
\]
\[
= 4 \sum_{i=1}^{3} \sum_{j=1}^{3} A_iB_j, \quad \text{since each of the sums} \sum_{i,j} A_iB_j, \sum_{i,j} B_jA_{i+1}, \sum_{i,j} B_iA_j, \sum_{i,j} A_jB_{i+1} \text{covers each of the 9 different} (A_i, B_j) \text{pairs exactly once and} A_iB_j = B_jA_i
\]
\[
= 4q.
\]

8. A rectangular chessboard has 5 rows and 2008 columns. Each square is painted either red or blue.

Determine the largest integer \( N \) which guarantees that, no matter how the chessboard is coloured, there are two rows which have matching colours in at least \( N \) columns.

**Solution.** Hint. Use the Pigeon Hole Principle to show \( N \geq 804 \). Then show \( N \leq 804 \). Finally, construct an example to show \( N = 804 \) is indeed the required solution.